Universität Regensburg

Department of Mathematics

Fakultät für Mathematik
Universitätsstraße 31
93053 Regensburg

Master Thesis

On the generating function and algebraicity of Eisenstein-Kronecker numbers

Sidi Mohammed Boughalem

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Supervised by
Prof. Dr. Guido Kings

Assistant Supervisor
Johannes Sprang

DEDICACES

In loving memory of John Horton Conway.

The “Moonshine” mathematician : 1937 - 2020.

I started writing this thesis during the COVID pandemic. As this pandemic took from each one of us something we used to take for granted, April,11 2020, it took from the world one of its remarkable geniuses. I would like to take this opportunity, and dedicate the merely meaningful efforts put in this work, to the loving memory of the man, who; through his Doomsday-algorithm, his Angel and Devil, his Sprouts; made me want to study mathematics.

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I hereby declare that the submitted printed copies of this thesis and the electronic version are identical, that I wrote this thesis myself with the help of no more than the mentioned literature and auxiliary means and that this thesis has not already been submitted to any other university for the award of a degree.

I also hereby declare being fully aware of the legal consequences provided for in section 26(6) of the examination regulations.

Regensburg, 16.11.2020

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(Signature Sidi Mohammed BOUGHALEM)

Abstract

It is well known that the Bernoulli numbers B_n hold a great arithmetical importance, due to their recurrent appearance in Number theory. In fact, Bernoulli numbers classically appear in power series expansions of trigonometric functions. They also are related to the Riemann ζ-function at negative integers.

One might also be familiar with the generalised Bernoulli numbers, which —in the same fashion— are related to the special values of the Dirichlet L-series. The Eisenstein-Kronecker numbers $e^*_{a,b}$ are an elliptic analogue of the generalised Bernoulli numbers, in the case of an imaginary quadratic field. They are, in this sense, related to the special values of Hecke L-series on imaginary quadratic fields.

In this thesis, we study the algebraic properties of the numbers $e^*_{a,b}$. We will follow Bannai and Kobayashi's paper [BK$^+$10] and show that the two-variable generating function of these numbers (which is the elliptic analogue of the cotangent function for the Bernoulli numbers) belongs to a canonical class of theta functions, called the reduced theta functions. We will then introduce Mumford's theory of algebraic theta functions in order to study the algebraic properties of $e^*_{a,b}$.

Zusammenfassung

Es ist bekannt, dass die Bernoulli-Zahlen B_n eine große arithmetische Bedeutung haben, da sie in der Zahlentheorie immer wieder vorkommen. Tatsächlich tauchen die Bernoulli-Zahlen klassischerweise in Potenzreihenentwicklungen trigonometrischer Funktionen auf. Sie sind auch mit der Riemann ζ-Funktion bei negativen ganzen Zahlen verwandt.

Man könnte auch mit den generalisierten Bernoulli-Zahlen vertraut sein, die — auf die gleiche Weise— mit den besonderen Werten der Dirichlet-L-Serie zusammenhängen. Die Eisenstein-Kronecker-Zahlen $e^*_{a,b}$ sind ein elliptisches Analogon zu den generalisierten Bernoulli-Zahlen, im Falle eines imaginären quadratischen Körpers. Sie sind in diesem Sinne mit den speziellen Werten der Hecke L-Reihe auf imaginären quadratischen Körpern verwandt.

In dieser Arbeit untersuchen wir die algebraischen Eigenschaften der Zahlen $e^*_{a,b}$. Wir werden der Arbeit [BK+10] von Bannai und Kobayashi folgen und zeigen, dass die Zwei-Variablen-Erzeugende Funktion dieser Zahlen (die das elliptische Analogon der Kotangensfunktion für die Bernoulli-Zahlen ist) zu einer kanonischen Klasse von Theta-Funktionen, den so genannten reduzierten Theta-Funktionen, gehört. Wir werden dann Mumfords Theorie der algebraischen Thetafunktionen vorstellen, um die algebraischen Eigenschaften von $e^*_{a,b}$ zu untersuchen.

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1 Introduction

"God exists since mathematics is consistent, and the Devil exists since we cannot prove it."

André Weil

1.1. Motivation

Let χ be a Dirichlet character of conductor $f_\chi = n$. Let $b \in \mathbb{N}$ such that

$$\chi(-k) = (-1)^b \chi(k).$$

Then, the Dirichlet L-series is given by

$$\begin{aligned} L(b,\chi) &:= \sum_{k=1}^{\infty} \frac{\chi(k)}{k^b} = \frac{1}{2} \sum_{k \in (\mathbb{Z}/n\mathbb{Z})^\times} \chi(k) \sum_{\substack{j \in 2\pi i \mathbb{Z} \\ j \neq -\frac{2\pi i}{n} k}} \frac{1}{\left(j \frac{n}{2\pi i} + k\right)^b} \\ &= \frac{(2\pi i)^b}{2n^b} \sum_{k \in (\mathbb{Z}/n\mathbb{Z})^\times} \chi(k) e_b^* \left(\frac{2\pi i}{n} k\right) \quad \text{where } e_b^* \left(\frac{2\pi i}{n} k\right) = \sum_{\substack{j \in 2\pi i \mathbb{Z} \\ j \neq -\frac{2\pi i}{n} k}} \frac{1}{\left(k + j \frac{2\pi i}{n} k\right)^b}. \end{aligned}$$

Now, for all $m \in \mathbb{Z}$, define the $2\pi i m$-periodic function:

$$\begin{aligned} G_m : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{e^z}{e^z - 1} \end{aligned}$$

Then, it is not hard to see that for all $z' \in \mathbb{C}$:

$$G_m(z + z') = \begin{cases} \frac{1}{z} + \sum_{b \geq 1} (-1)^{b-1} z^{b-1} e_b^*(z') & \text{if } z' = 2\pi i m \\ \sum_{b \geq 1} (-1)^{b-1} z^{b-1} e_b^*(z') & \text{otherwise.} \end{cases}$$

G_m is said to be a *generating function* for the numbers e_b^*. Observe that this is not a rational function. However, algebraically, one has an isomorphism of algebraic groups

$$\begin{aligned} \mathbb{C}/_{2\pi i \mathbb{Z}} &\xrightarrow{\sim} \mathbb{C}^\times \\ z &\longmapsto T = e^z \end{aligned}$$

Now, if $z' = q \in 2\pi i\mathbb{Z} \otimes \mathbb{Q}$, then under the above isomorphism, the generating function $G_m(z+q)$ near $z = 0$ corresponds to a rational function

$$\frac{t}{t-1}$$

Thus, by choosing an embedding $i : \overline{Q} \hookrightarrow \mathbb{C}$, one gets that the numbers $e_b^*(q) \in \overline{Q}$ are algebraic.

1.2. Objective

The idea of Bannai and Kobayashi in [BK⁺10] was to reproduce the same reasoning for an imaginary quadratic field, i.e. define a two variable generating function for the Eisenstein-Kronecker numbers $e_{a,b}^*$, which are special values of Hecke L-series attached to a Hecke character χ on an imaginary quadratic field K.

Given an elliptic curve $E(\mathbb{C})$ with CM by the ring of integers of an imaginary quadratic field K, the generating function of the Eisenstein-Kronecker numbers $e_{a,b}^*$ is a meromorphic function Θ, known as the Kronecker theta function. We will show that this generating function is a *canonical* meromorphic section of the Poincaré bundle of an elliptic curve, then; under complex multiplication; that its Laurent series has rational coefficients. This shows the algebraicity of the Eisenstein-Kronecker numbers.

1.3. Scope

The subtlety of this approach resides in the following observation: remark that in §1.1, one considers the algebraic model of $\mathbb{C}/_{2\pi i\mathbb{Z}}$, and under this uniformisation, associates a rational function to the generating function. Then, one would have to consider a translation operator that would preserve the algebraicity; an *algebraic* translation. This is done through Mumford's theory of algebraic theta function, for general theta functions on an algebraic variety $A(k)$; over a general ground field k.

Let V be a g-dimensional complex vector space. Mumford's theory of algebraic theta functions allows (among other things) the study of sections s of line bundles at any torsion point. The translation operator in §4.1.10 preserves the reducedness of the theta functions, and; through the isomorphism (4.3)

$$t_w^*\mathcal{L} \cong \mathcal{L}(H, \chi \cdot \alpha_w);$$

determines the reduced theta function corresponding to the section $t_w^* s$ up to a constant multiple. This constant however, depends on the choice of the above isomorphism. Moreover, the translation operator considered in §4.1.10 also depends on a semi-character $\widetilde{\chi} : V \longrightarrow \mathbb{C}$. Mumford's theory allows an *algebraic* construction of the translation operator that canonically determines such an isomorphism, which in turn, determines the reduced theta function corresponding to the section $t_w^* s$ up to an n-th root of unity.

Finally, Figure 1.1 attempts to show "*the big picture*" of the thesis:

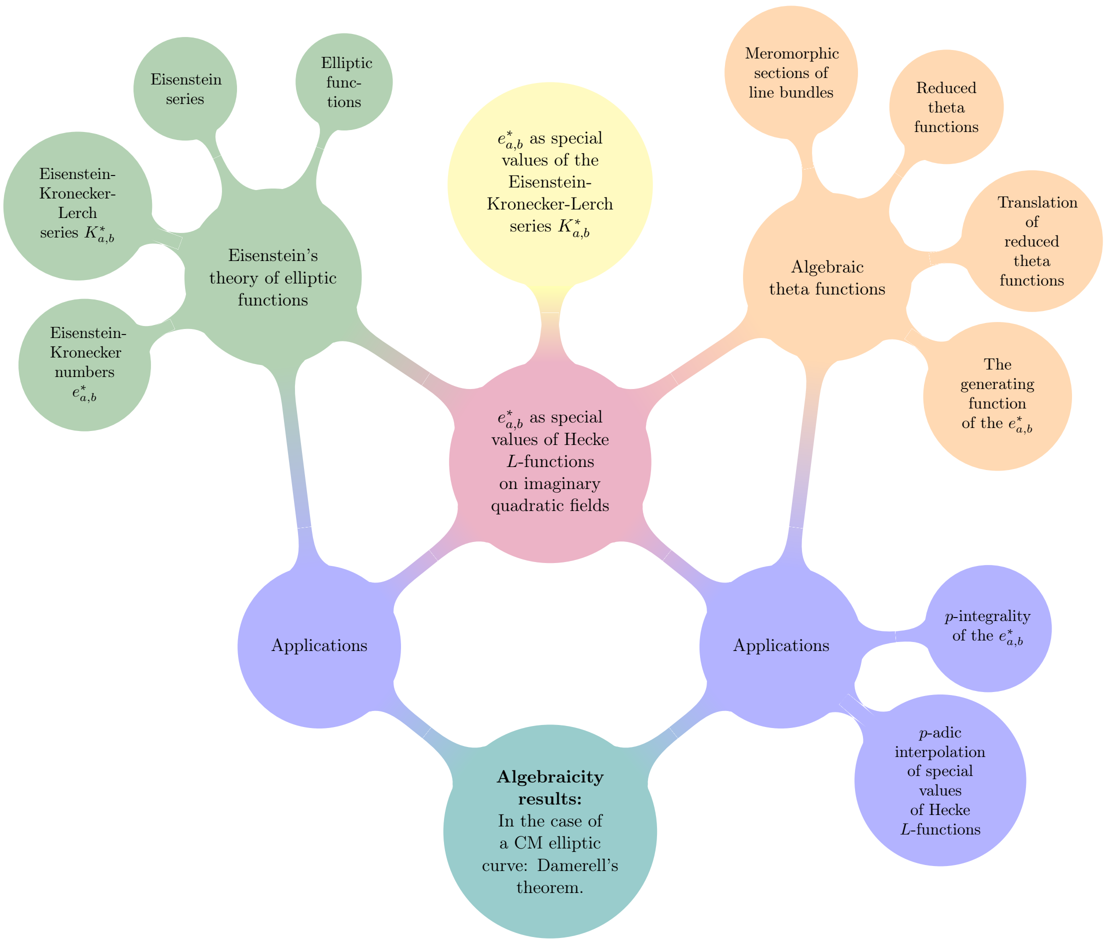

Figure 1.1.: Conceptual summary of the thesis

1.4. Outline

This thesis is separated into 5 chapters, with two main parts. Each chapter provides applications and general results to illustrate the work done, and give some additional context and insight.

In **Chapter 2**, we introduce the Eisenstein-Kronecker series from elliptic functions, which have been thoroughly studied by Eisenstein and Kronecker in the late 19th century. The main source of this section is the very complete book from Prof. André Weil [Wei76]. We start by introducing Eisenstein's one variable "trigonometric functions" and use his methods to define the Eisenstein-Kronecker-Lerch series. We relate the most relevant content with modern terminology and introduce in the end the Eisenstein-Kronecker numbers. The main result of this section is proposition 2.3.3 which shows analytic continuation and the functional equation of the Eisenstein-Kronecker-Lerch K_a^*. This is a crucial part in the proof of the main theorem (theorem 4.3.1) of this thesis.

In **Chapter 3**, we introduce Hecke characters: L-functions usually arise as the analytic continuation of an L-series; first defined as a Dirichlet series, then expanded into an Euler product indexed by primes. Hence, Hecke characters were once introduced by Hecke in order to establish the analytic continuation (and functional equation) of a Dirichlet series for a general number field K. We then introduce *algebraic* Hecke characters, which are mainly motivated by complex multiplication. Here is a modest and rough motivation: The main theorem of complex multiplication for an abelian variety A (theorem 3.3.5 in the case of an elliptic curve); when restated in Galois theoretic formulation; is nicely tied to the l-adic representations of $\mathrm{Gal}(K^{ab}/K)$ on the l-adic Tate module $T_l(A)$. This motivates the idèlic formulation of Hecke characters and suggests that such objects should be restrictions of some homomorphisms of algebraic groups. (§7 of the very excellent paper [ST68] provides a more complete and detailed motivation). The main theorem of this section is theorem 3.2.12 where we relate the Hecke L-series to the Eisenstein-Kronecker-Lerch series K_a^*. In the case of a CM Elliptic curve, we use complex multiplication and Deuring's theorem B.3.10 to relate the special values of the L-function attached to a CM elliptic curve to the Eisenstein-Kronecker numbers $e_{a,b}^*$ through the functional equation and analytic continuity of the Hecke L-series. We end the section by presenting the two main conjectures related to special values of L-function of elliptic curves.

Chapter 4 is somehow the core of the thesis. We introduce here a different approach to study the special values of Hecke L-functions on imaginary quadratic fields, through the theory of theta functions. We review the theory of line bundles and *reduced* theta functions (which are, in a sense, some sort of canonical theta functions) over abelian varieties. Then we define the Kronecker Theta function Θ as a reduced theta function associated to the Poincaré bundle of an elliptic curve, which is nothing but a line bundle on the abelian variety $E \times E^\vee$ (this is possible because elliptic curves -under complex uniformisation- are self dual). We define a theta-translation operator that preserves the *reducedness* and show that (theorem 4.3.1): under this translation, the Kronecker theta function $\Theta_{w,w'}(z,z')$ is a two-variable generating function of the Eisenstein-Kronecker numbers $e_{a,b}^*(w,w')$. The main ingredient in the proof of this theorem is Kronecker's theorem 4.2.10, which relates (in the case of an elliptic curve) the Kronecker theta function $\Theta(z,z')$ to the Eisenstein-Kronecker-Lerch series $K_1(z,z',1)$.

Chapter 5 is devoted to the study of the algebraicity of the Eisenstein-Kronecker numbers. Being related to critical values of Hecke L-functions on imaginary quadratic fields, Damerell proved their algebraicity in [Dam70]. We adapt his proof to our setting and present his result (theorem 5.1.1) in the first part of this chapter. In the second, we introduce Mumford's theory of algebraic theta functions. We first give an overview of the general theory for abelian varieties over a general field. Mumford's theory allows an *algebraic* construction of the translation operator that; given a section s_D; canonically determines the reduced theta function corresponding to its translation $t_w^* s$ up to an n-th root of unity, while preserving the algebraicity of the coefficients. Applied in the case $A = E \times E^\vee$; where $E(\mathbb{C})$ is a CM elliptic curve; the generating function of the Eisenstein-Kronecker numbers $e^*_{a,b}(w, w')$ is a reduced theta function Θ, with an algebraic divisor. Thus, Mumford's theory shows that; for rational torsion points $w, w' \in \Gamma \otimes \mathbb{Q}$; the numbers $e^*_{a,b}(w, w')$ are algebraic.

Appendix A and **B** provide additional background about most of the arguments used in the four chapters. **Appendix A** presents most of the analytical tools used in chapter 2 (Poisson summation, Fourier analysis on the torus, convergence results...) and some additional results on elliptic functions. **Appendix B** provides a relatively sufficient algebraic background and interpretation of results treated in the thesis. The first part reviews most of the implicit results about class field theory used in chapters 3, 4 and 5 (most of the results are taken from the excellent lecture notes [Mil08b]), the second describes abelian varieties algebraically (here again, we solely rely on Mumford's book [DM70]. Some proofs/results were propositions/exercises I have done for my previous courses of Algebraic Geometry). The last part gives a general framework on Elliptic curves, that helps understanding most of the results in chapter 3 and chapter 5.

2

Eisenstein and Kronecker's theory of elliptic functions

This section is intended to give a relatively detailed introduction about the study of the Eisenstein-Kronecker-Lerch. Most of this work is marvellously done in [Wei76].

2.1. Eisenstein's trigonometric functions

Let us consider the series

$$\varepsilon_n(z) := \sum_{k=-\infty}^{+\infty} \frac{1}{(z+k)^n}$$

Where $n \in \mathbb{N}_{>0}$. Let us at first consider the case where $n \geq 2$: We first rewrite the series as the following

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(z+k)^n} = \sum_{k=1}^{+\infty} \frac{1}{(z-k)^n} + \sum_{k=1}^{+\infty} \frac{1}{(z+k)^n} + \frac{1}{z^n}$$

We treat the first series on the LHS, the other one is similar. Let $z \in \mathbb{C} \smallsetminus \mathbb{Z}$, there exists an $N \in \mathbb{N}$ such that $|z| < N - 1$. We consider the compact sets $K_N := \{z \in \mathbb{C}, |z| \leq N\} \smallsetminus \mathbb{N}$. Then for $k \geq 2N$ and a fixed $z \in K_N$ one has

$$|z| \leq N \leq \frac{k}{2} \quad \Rightarrow \quad |z-k| \geq k - |z| \geq k - \frac{k}{2} = \frac{k}{2}$$

Hence one gets

$$\frac{1}{|z-k|^n} \leq \frac{2^n}{k^n}$$

The above series is absolutely convergent on each compact set that does not contain an integer, thus $\varepsilon_n(z)$ converges normally to meromorphic functions for $n \geq 2$.

For $n = 1$, we consider a special method of summation, namely

$$\varepsilon_1(z) = \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{z+k} \tag{2.1}$$

Now observe that

$$\sum_{k=-N}^{N} \frac{1}{(z+k)} = \frac{1}{z} + \sum_{k=1}^{N} \frac{1}{(z+k)} + \frac{1}{(z-k)} = \frac{1}{z} + 2z \sum_{k=1}^{N} \frac{1}{(z^2+k^2)}$$

On the other hand, for a fixed $z \in K_N$

$$\left|\frac{1}{(z^2+k^2)}\right| \leq \frac{4}{k^2}$$

which converges absolutely. Thus, by the same reasoning as above, we get normal convergence of the series to a meromorphic function $\varepsilon_1(z)$.

Now uniform convergence allows taking term-by-term derivatives, this shows that for all $n \geq 1$

$$\frac{d}{dz}\varepsilon_n = -n\varepsilon_{n+1}(z) \tag{2.2}$$

And one sees that

$$\begin{aligned}
\varepsilon_1(z+1) &= \frac{1}{z+1} + \lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{1}{z+1+k} + \frac{1}{z+1-k}\right) \\
&= \frac{1}{z+1} + \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{z+1+k} + \sum_{k=1}^{N}\frac{1}{z+-k+1}\right) \\
&= \frac{1}{z+1} + \lim_{N\to\infty}\left(\sum_{k=2}^{N+1}\frac{1}{z+k} + \sum_{k=0}^{N-1}\frac{1}{z-k}\right) \\
&= \frac{1}{z+1} + \lim_{N\to\infty}\left(\frac{1}{z} - \frac{1}{z+1} + \frac{1}{z+N+1} - \frac{1}{z-N} + \sum_{k=1}^{N}\left(\frac{1}{z+k} + \frac{1}{z-k}\right)\right) \\
&= \frac{1}{z} + \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{z+k} + \frac{1}{z-k}\right) = \varepsilon_1(z)
\end{aligned}$$

Hence ε_n is 1-periodic for all $n \geq 1$. Recall that the cotangent function $\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$ is an odd, meromorphic function, with countably many poles (at each $m \in \mathbb{Z}$). From the power series expansion of the sine and cosine functions, one clearly sees that

$$\cot(\pi z) = \frac{1}{\pi(z-n)} + \sum_m a_m(z-n)^m$$

Moreover, it satisfies the well known trigonometric formula (called the double angle formula)

$$\cot(2\theta) = \frac{\cot(\theta)^2 - 1}{2\cot(\theta)} \tag{2.3}$$

Now, let $g(z) := \pi\cot(\pi z)$, and let f be any other odd meromorphic function with principal part $\frac{1}{(z-m)}$ at each $m \in \mathbb{Z}$. Then the function $h(z) := f(z) - g(z)$ is odd and satisfies

$$\begin{cases} h(z) = 2h(2z) - h\left(\frac{1+2z}{2}\right) \\ h(0) = 0 \end{cases} \tag{2.4}$$

By the maximum principle theorem, there exists a $z_0 \in B(0,2)$ such that $|h(z)| < |h(z_0)|$ for some $z \in B(0,2)$. In particular,

$$\left|h\left(\frac{z_0}{2}\right) + h\left(\frac{1+z_0}{2}\right)\right| \leq \left|h\left(\frac{z_0}{2}\right)\right| + \left|h\left(\frac{1+z_0}{2}\right)\right| < 2|h(z_0)|$$

which contradicts (2.4). Hence, h must be identically zero and the equation (2.3) completely characterises the cotangent function. Now as seen in (2.1), ϵ_1 is odd and meromorphic with principal part $\frac{1}{z-m}$ at each integer m. Moreover, if we denote by S_N the partial sum of ε_1 then

$$\begin{aligned}
S_k(z) + S_k\left(z + \frac{1}{2}\right) &= \frac{1}{z} + \frac{1}{z+\frac{1}{2}} + \sum_{k=1}^{N} \frac{1}{(z+k)} + \frac{1}{(z-k)} + \sum_{k=1}^{N} \frac{1}{(z+\frac{1}{2}+k)} + \frac{1}{(z+\frac{1}{2}-k)} \\
&= \frac{1}{z} + \frac{2}{2z+1} + \sum_{k=2}^{2N} \frac{2}{(2z+1+k)} + \sum_{k=2}^{2N} \frac{2}{(2z+1-k)} \\
&= \frac{2}{z} + \frac{2}{2z+2N+1} + 2\sum_{k=1}^{2N} \frac{1}{(2z+k)} - \frac{1}{(2z-k)} = 2S_{2N}(2z) + \frac{2}{2z+2N+1}
\end{aligned}$$

This implies that

$$\varepsilon_1(z) + \varepsilon_1\left(z + \frac{1}{2}\right) = 2\varepsilon_1(2z)$$

And finally

$$\varepsilon_1(z) = \pi \cot(\pi z).$$

Hence, from the double angle formula (2.3) one easily derives an addition formula for ε_1, namely

$$\varepsilon_1(z+w) = \frac{\varepsilon_1(z)\varepsilon_1(w) - \pi^2}{\varepsilon_1(z) + \varepsilon_1(w)} \tag{2.5}$$

One deduces several known partial fraction series from ε_1 by (2.2) as follows:

$$\begin{aligned}
\varepsilon_2(z) &= \sum_{k\in\mathbb{Z}} \frac{1}{(z+k)^2} = \frac{\pi^2}{\sin^2 \pi z} \\
\varepsilon_3(z) &= \sum_{k\in\mathbb{Z}} \frac{1}{(z+k)^3} = \pi^3 \frac{\cot \pi z}{\sin^2 \pi z} \\
&= \varepsilon_1(z)\varepsilon_2(z)
\end{aligned} \tag{2.6}$$

As well as the known identify (easily deduced from (2.3))

$$\begin{aligned}
\frac{\pi}{\sin \pi z} &= \pi \cot \pi z + \pi \tan \frac{\pi z}{2} = \pi \cot \pi z + \pi \cot \frac{\pi z}{2} - 2\pi \cot \pi z \\
&= \varepsilon_1(z) + \sum_{k=0}^{\infty} \frac{4z}{(2k+1)^2 - z^2} = \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k \frac{2z}{z^2 - k^2}
\end{aligned}$$

Or more elegantly

$$\frac{\pi}{\sin \pi z} = \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{z+k}$$

For further use, we will develop a slightly more general formula from (2.6) as follows: First, notice that from the differential equation $(\cot z)' = -1 - \cot(z)^2$ and using the fact $\epsilon_2 = -\epsilon_1'$ from (2.2), one deduces that

$$\epsilon_2(z) = \epsilon_1^2(z) + \pi^2 \tag{2.7}$$

Now, by (2.6)

$$\begin{aligned}
\epsilon_3(z+w) = \epsilon_2(z+w)\epsilon_1(z+w) &= \epsilon_2(z+w)\frac{\varepsilon_1(z)\varepsilon_1(w) - \pi^2}{\varepsilon_1(z) + \varepsilon_1(w)} \\
\Leftrightarrow 2\epsilon_3(z+w)(\varepsilon_1(z) + \varepsilon_1(w)) &= \epsilon_2(z+w)\left(2\varepsilon_1(z)\varepsilon_1(w) - 2\pi^2\right) \\
&= \epsilon_2(z+w)\left(2\varepsilon_1(z)\varepsilon_1(w) + \epsilon_1(z)^2 - \epsilon_2(z) + \epsilon_1(w)^2 - \epsilon_2(w)\right) \\
&= \epsilon_2(z+w)\left(\epsilon_1(z) + \epsilon_1(w)\right)^2 - \epsilon_1(z)\epsilon_2(z+w) - \epsilon_1(w)\epsilon_2(z+w)
\end{aligned}$$

Hence, by combining (2.5) and (2.7) once more, we finally get the addition formula

$$2\epsilon_3(z+w)(\varepsilon_1(z) + \varepsilon_1(w)) = \epsilon_2(z)\epsilon_2(w) - \epsilon_1(z)\epsilon_2(z+w) - \epsilon_1(w)\epsilon_2(z+w) \tag{2.8}$$

We conclude this section by another interesting expression of ε_1 that can be obtained as follows: Consider the function $f(z) = (z^2 - k^2)^{-1}$ on the unit disk, given by the following power series expansion

$$\frac{2}{z^2 - k^2} = -\frac{2}{k^2}\sum_{n=0}^{\infty}\left(\frac{z}{k}\right)^{2n}$$

Then, the $(2n-2)$th term of its Taylor series is $\frac{-2}{k^{2n}}$. Since the series $\sum\limits_{k\geq 1} 2(z^2 - k^2)^{-1}$ converges uniformly on the unit disk, its $(2n-2)$th Taylor coefficient is nothing but

$$-2\sum_{k\geq 1}\frac{1}{k^{2n}} = 2\zeta(2n)$$

Now from the Taylor expansion of the cotangent function at 0, one gets that

$$\begin{aligned}
\varepsilon_1(z) &= \frac{1}{z} - \sum_{k=1}^{\infty} q_{2k} z^{2k-1}, \quad \text{where } z \in \mathbb{S}^1 \smallsetminus \{0\}, \quad q_{2k} = 2\zeta(2k) \qquad (2.9)\\
&= \frac{1}{z} - \sum_{k=1}^{\infty}\frac{(2\pi)^{2k}}{(2k!)}B_{2k}z^{2k-1}
\end{aligned}$$

Differentiating n times, we get the power series expansion of ε_n

$$\varepsilon_n(z) - \frac{1}{z^n} = (-1)^n\sum_{k=1}^{\infty}\binom{2k-1}{n-1}\frac{(2\pi)^{2k}}{(2k!)}B_{2k} \tag{2.10}$$

In particular, comparing coefficient near $z = 0$ in (2.9) we obtain for $n \geq 1$ the famous identity:

$$\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n!)}B_{2n}$$

2.2. Eisenstein's elliptic functions

In the following, we consider two complex variables. In particular, consider a fundamental pair of periods ω_1, ω_2, i.e. non-zero complex numbers such that $\Im(\tau) > 0, \tau := \frac{\omega_1}{\omega_2}$. They define a lattice in $\mathbb{C}$ that will be denote $\Gamma = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$.

2.2.1. Eisenstein series

Definition 2.2.2 (Eisenstein series)**.** Let $w \in \Gamma, w = m_1\omega_1 + m_2\omega_2$. For a positive integer $n > 0$, we define the **Eisenstein series** to be the double series

$$E_n(z, \omega_1, \omega_2) := \sum_{m_1, m_2}{}_e \frac{1}{(z + m_1\omega_1 + m_2\omega_2)^k} \tag{2.11}$$

where $\sum_e$ is known as the **Eisenstein summation** and is defined as

$$\sum_{m_1,m_2}{}_e = \sum_{m_2}{}_e \left(\sum_{m_1}{}_e \right) = \lim_{N\to\infty} \sum_{m_2=-N}^{N} \left(\lim_{M\to\infty} \sum_{m_1=-M}^{M} \right)$$

The series $E_n(z, \omega_1, \omega_2)$ converges normally on the domains $D := \mathbb{C} \setminus \bigcup_{m_1,m_2} (-m_1\omega_1 - m_2\omega_2)$ for $n \geq 3$. Indeed, let us fix some z in some compact set inside D.
For $w \in \Gamma \cap \mathbb{C} \setminus B(0, \frac{|z|}{2})$, one has $|w| \geq \frac{|z|}{2}$, thus

$$\frac{1}{|w + z|^k} \leq \frac{1}{||w| - |z||^k} \leq \frac{1}{(1 - \frac{1}{2})^k |w|^k} = \frac{2^k}{|w|^k}$$

The result follows from Lemma A.1.1.

Now for $k \geq 3$, the Eisenstein summation is not needed, as it coincides with the regular sum over the lattice Γ, namely

$$\begin{aligned} E_n(z, \omega_1, \omega_2) = \sum_{m_1,m_2}{}_e \frac{1}{(z + m_1\omega_1 + m_2\omega_2)^k} &= \sum_{m_1,m_2=-\infty}^{\infty} \frac{1}{(z + m_1\omega_1 + m_2\omega_2)^k} \\ &= \sum_{w\in\Gamma} \frac{1}{(z + w)^k} =: E_n(z, \Gamma) \end{aligned}$$

Moreover, E_n is Γ-periodic, with the period lattice (ω_1, ω_2). Indeed, on has for all integers $a, b \in \mathbb{Z}$

$$E_n(z + a\omega_1 + b\omega_2, \omega_1, \omega_2) = E_n(z, \omega_1, \omega_2)$$

For the case where $n = 1, 2$ a little bit of work is needed. If u, v are two other generators of Γ, i.e. $w = \mu u + \nu v = m_1\omega_1 + m_2\omega_2$, then the summation can be made more explicit by using ε_n. Pose

$$\xi = \frac{z}{u} \quad , \quad \tau = \frac{v}{u}$$

then one sees that

$$\sum_{\mu}^{e} \frac{1}{z+\mu u+\nu v} = \sum_{\mu}^{e} \frac{1}{(u^n)(\frac{z}{u}+\frac{\nu v}{u}+\mu)^n} = u^{-n}\varepsilon_n\left(\frac{z+\nu v}{u}\right)$$

hence equation (2.11) becomes

$$E_n(z,u,v) = u^{-n}\sum_{\nu}^{e}\varepsilon_n\left(\frac{z+\nu v}{u}\right) = u^{-n}\sum_{\nu}^{e}\varepsilon_n(\xi+\nu\tau). \tag{2.12}$$

- We start by the case $n=2$:

$$\sum^{e}(z+w)^{-2} = u^{-2}\sum_{\nu}^{e}\varepsilon_2(\xi+\nu\tau)$$

Recall that from (2.2)

$$\varepsilon_n(\xi) = -\frac{1}{n-1}\frac{d}{dz}\varepsilon_{n-1}(\xi) = \frac{1}{(n-1)(n-2)}\left(\frac{d}{d\xi}\right)^2\varepsilon_{n-2}(\xi) = \cdots = \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{d}{d\xi}\right)^{n-1}\varepsilon_1(\xi)$$

One also has from the previous section that

$$\varepsilon_1(z) = \pi\cot\pi z = \pi\frac{\cos\pi z}{\sin\pi z} = \pi\frac{e^{iz}+e^{-iz}}{2}\cdot\frac{2i}{e^{iz}-e^{-iz}} = \pi i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$$

Hence, applying the substitution from above, and by putting $\widehat{z} = e^{2\pi i\xi}, q = e^{i\pi\tau}$, one gets

$$\varepsilon_1(\xi+\nu\tau) = \begin{cases} \pi i\left(\frac{q^{2\nu}\widehat{z}+1}{q^{2\nu}\widehat{z}-1}\right) = \pi i - 2\pi i\left(\frac{1}{1-q^{2\nu}\widehat{z}}\right) & \text{if } \nu>0 \\ \\ \pi i\left(\frac{q^{-2\nu}\widehat{z}+1}{q^{-2\nu}\widehat{z}-1}\right) = -\pi i + 2\pi i\left(\frac{1}{1-q^{2\nu}\widehat{z}^{-1}}\right) & \text{if } \nu<0 \end{cases} \tag{2.13}$$

and by (2.2)

$$\varepsilon_n(\xi+\tau\nu) = \begin{cases} \frac{(-2\pi i)^n}{(n-1)!}\left(\widehat{z}\frac{d}{d\widehat{z}}\right)^{n-1}\left(\frac{1}{1-q^{2\nu}\widehat{z}}\right) & \text{if } \nu>0 \\ \\ \frac{(2\pi i)^n}{(n-1)!}\left(\widehat{z}\frac{d}{d\widehat{z}}\right)^{n-1}\left(\frac{1}{1-q^{2\nu}\widehat{z}^{-1}}\right) & \text{if } \nu<0 \end{cases}$$

Now, since $|q|<1$ and for large enough $\nu>0$ such that $|q^{2\nu}\widehat{z}|<1$ we get

$$\varepsilon_n(\xi+\tau\nu) = \frac{(-2\pi i)^n}{(n-1)!}\left(\widehat{z}\frac{d}{d\widehat{z}}\right)^{n-1}\left(\sum_{k=1}^{\infty}q^{2\nu k}\widehat{z}^k\right) = \frac{(-2\pi i)^n}{(n-1)!}\sum_{k=1}^{\infty}k^{n-1}q^{2\nu k}\widehat{z}^k$$

and taking the absolute value, one gets

$$|\varepsilon_n(\xi+\tau\nu)| \le C(\widehat{z})|q|^{2\nu} \qquad \text{where } C(\widehat{z}) \text{ does not depend on } \nu.$$

Similarly, for $\nu<0$, replacing by $-\nu$ one gets for large enough ν such that $|q^{2\nu}\widehat{z}^{-1}|<1$

$$\varepsilon_n(\xi-\tau\nu) = \frac{(2\pi i)^n}{(n-1)!}\left(\widehat{z}\frac{d}{d\widehat{z}}\right)^{n-1}\left(\sum_{k=1}^{\infty}q^{2\nu k}\widehat{z}^k\right) = \frac{(-2\pi i)^n}{(n-1)!}\sum_{k=1}^{\infty}k^{n-1}q^{2\nu k}\widehat{z}^{-k}.$$

Now for all $z \in K \subset \{z \in \mathbb{C} \smallsetminus \mathbb{Z} \mid |q| < \widehat{z} < \frac{1}{|q|}\}$, the following series becomes

$$E_n(z,u,v) = u^{-n}\varepsilon_n(\xi) + \frac{(-1)^n}{(n-1)!}\left(\frac{2\pi i}{u}\right)^n \sum_{\nu=1}^{\infty}\sum_{k=1}^{\infty} k^{n-1}q^{2\nu k}(\widehat{z}^k + (-1)^n\widehat{z}^{-k}) \quad (2.14)$$

which converges absolutely and uniformly for $n \geq 2$ inside each K.

- For $n = 1$:

$$\sum\nolimits_e (z+w)^{-1} = u^{-1}\sum_{\nu}\nolimits_e \varepsilon_1(\xi+\nu\tau) = u^{-1}\varepsilon_1(\xi) + u^{-1}\sum_{\nu=1}^{+\infty}\varepsilon_1(\xi+\nu\tau) + \varepsilon_1(\xi-\nu\tau)$$

Hence, by adopting the same notations as above once again, one gets

$$\begin{aligned}\varepsilon_1(\xi+\nu\tau)+\varepsilon_1(\xi-\nu\tau) &= \pi i\left(\frac{q^{2\nu}\widehat{z}+1}{q^{2\nu}\widehat{z}-1} + \frac{q^{-2\nu}\widehat{z}+1}{q^{-2\nu}\widehat{z}-1}\right)\\ &= -2\pi i\left(\frac{1}{1-q^{2\nu}\widehat{z}} + \frac{1}{q^{2\nu}\widehat{z}^{-1}}\right)\end{aligned}$$

and by the same reasoning, for large ν

$$\left|\frac{1}{1-q^{2\nu}\widehat{z}} + \frac{1}{q^{2\nu}\widehat{z}^{-1}}\right| \leq \left|\frac{1}{1-|q^{2\nu}\widehat{z}|}\right| + \left|\frac{1}{1-|q^{2\nu}\widehat{z}^{-1}|}\right| \leq C(\widehat{z})|q|^{2\nu}$$

which proves convergence for all $n \geq 1$.

2.2.3. Power series expansion

The nature of convergence of the Eisenstein series allows us in particular to take term-by-term derivatives and one easily obtains the very useful identity

$$\frac{d}{dz}E_n = -nE_{n+1} \quad (2.15)$$

One can (as in the case of trigonometric functions) use (2.15) to get a power series expansion of E_n. To do so, observe first that $E_1(z,u,v)$ is an odd function in z. One sees that

$$\begin{aligned}E_1(z,u,v) = \frac{E_1(z,u,v)-E_1(-z,u,v)}{2} &= \frac{1}{z} + \frac{1}{2}\sum_{\mu,\nu\neq 0}\nolimits_e \frac{1}{z+\mu u+\nu v} - \frac{1}{-z+\mu u+\nu v}\\ &= \frac{1}{z} + \frac{1}{2}\sum_{\mu,\nu\neq 0}\nolimits_e \frac{1}{z+w} + \frac{1}{z-w}\end{aligned}$$

Taking $|z| < |\mu u + \nu v| = |w|$ for all $\mu,\nu \in \mathbb{Z}^*$, and expanding $(z+w)^{-1}$ and $(z-w)^{-1}$ into power series in z, one sees that the terms with odd index cancel and

$$E_1(z,u,v) = \frac{1}{z} - \sum_{\mu,\nu\neq 0}\nolimits_e \left(\sum_{k=1}^{\infty}\frac{z^{2k-1}}{w^{2k}}\right)$$

Now, the power series being absolutely convergent, one writes

$$E_1(z,u,v) = \frac{1}{z} - \sum_{k=1}^{\infty}\left(\sum\nolimits_{\mu,\nu\neq 0}^{e} \frac{1}{w^{2k}}\right) z^{2k-1}$$

Put

$$E_1(z,u,v) - \frac{1}{z} = -\sum_{k=1}^{\infty} e_{2k} z^{2k-1}$$

where

$$\begin{cases} e_2 = \sum\nolimits_{\mu,\nu\neq 0}^{e} \frac{1}{w^2} & k = 1 \\ e_{2k} = \sum\limits_{\mu,\nu\neq 0} \frac{1}{w^{2k}} & k \geq 2 \\ e_{2k+1} = 0 & k \geq 1 \end{cases}$$

By differentiating n times, one gets the power series expansion for E_n

$$E_n(z,u,v) - \frac{1}{z^n} = (-1)^n \sum_{k=1}^{\infty} \binom{2k-1}{n-1} e_{2k} z^{2k-n} \tag{2.16}$$

where e_{2k} is the value of $E_{2k}(z,u,v) - \frac{1}{z^{2k}}$ near 0.

Observe that, from (2.10) and adopting the notation from §2.2.1, the $2k$-th coefficient of $\varepsilon_{2k}(\xi) - \frac{1}{\xi^{2k}}$ near 0 is

$$q_{2k} = \frac{(2\pi)^{2k}}{(2k)!} B_{2k}$$

Thus, near $z = 0$ ($\xi = 0$ and $\widehat{z} = 1$), one gets by (2.14)

$$\begin{aligned} e_{2k} &= u^{-2k}\left(\varepsilon_{2k}(\xi) - \frac{1}{\xi^{2k}}\right) + 2u^{-2k}\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{d=1}^{\infty} d^{2k-1} q^{2d} \sum_{\nu=0}^{\infty} (q^{2d})^{\nu} \\ &= \frac{(-1)^k}{(2k)!}\left(\frac{2\pi i}{u}\right)^{2k} B_{2k} + \frac{2}{(2k-1)!}\left(\frac{2\pi i}{u}\right)^{2k} \sum_{d=1}^{\infty} d^{2k-1} q^{2d}(1 + q^{2d} + q^{4d} + \dots) \end{aligned}$$

and finally

$$e_{2k} = \frac{2}{(2k-1)!}\left(\frac{2\pi i}{u}\right)^{2k}\left(\frac{(-1)^k}{4k} B_{2k} + \sum_{d=1}^{\infty} \sigma_{2k-1}(d) q^{2d}\right)$$

where

$$\sigma_{2k-1}(d) = \sum_{d|\nu} d^{2k-1}.$$

Remark 2.2.4. In a more modern terminology, one might write

$$E_2(z,u,v) - e_2 = \frac{1}{z^2} + \sum\nolimits_{\mu,\nu\neq 0}^{e} \frac{1}{(z+\mu u+\nu v)^2} - \frac{1}{(\mu u+\nu v)^2}$$

By absolute convergence (2.14) one does not need the Eisenstein summation anymore, thus one recovers the usual Weierstrass's functions

$$E_2(z,w) - e_2 = \frac{1}{z^2} + \sum_{\gamma\in\Gamma\smallsetminus\{0\}} \left(\frac{1}{(z+\gamma)^2} - \frac{1}{\gamma^2}\right) := \wp(z,\Gamma)$$

And similarly, by convergence argument

$$E_1(z,w) - e_2 z = \frac{1}{z} + \sum_{\gamma\in\Gamma\smallsetminus\{0\}} \left(\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2}\right) := \zeta(z,\Gamma) \tag{2.17}$$

2.2.5. Lattice dependence

As it was shown in §2.2.1, for $n \geq 3$, the summation process depends only on the lattice Γ. But for $n = 1,2$, the summation process depends also on the generators of the lattice. For example:

(i) Interchanging the order of the double summation i.e. interchanging the roles of μ and ν amounts to changing the generators u, v.

(ii) Making a translation on a lattice, say $z + w_0$ in E_n amounts to substituting, respectively, μ, ν by $\mu + \mu_0, \nu + \nu_0$ which by itself changes the generators u, v.

Thus it is only natural to expect that any changes on the generators in the case, where one uses a different summation process (i.e. $n = 1,2$) would produce a different summation. We will treat the case that would be most relevant for the topic's study, which is the summation (that we will note $\sum_e{}'$) obtained from $\sum_e$ by a translation on Γ (which amounts to studying the periodicity of E_n with respect to Γ). We thus denote by E'_n the function obtained by applying $\sum_e{}'$.

It is clear that for $n \geq 3$, $E_n = E'_n$ since the series converges absolutely. For $n = 1,2$ From (2.15) we have

$$\frac{d}{dz}E'_1(z,u',v') = -E'_2(z,u',v') \qquad \frac{d}{dz}E'_2(z,u',v') = -2E'_3(z,u',v') = -2E_3(z,u,v)$$

Hence

$$\begin{cases} E'_2(z,u',v') - E_2(z,u,v) = & A(u,v) \\ E'_1(z,u',v') - E_1(z,u,v) = & A(u,v)z + B(u,v) \end{cases} \tag{2.18}$$

Now $E'_n(z,u',v') = E_n(z+w_0,u,v)$, where $w_0 = mu + m'v$. At first, notice that since ε_1 is 1-periodic, then for all integer a

$$\begin{aligned} E_1(z+au,u,v) &= u^{-1}\sum_{\nu}{}_e\, \varepsilon_1(\xi+\nu\tau) = u^{-1}\varepsilon_1(\xi+a) + u^{-1}\sum_{\nu=1}^{+\infty}\varepsilon_1(\xi+a+\nu\tau) + \varepsilon_1(\xi+a-\nu\tau) \\ &= u^{-1}\varepsilon_1(\xi) + u^{-1}\sum_{\nu=1}^{+\infty}\varepsilon_1(\xi+\nu\tau) + \varepsilon_1(\xi-\nu\tau) = E_1(z,u,v) \end{aligned}$$

By the same reasoning,

$$E_2(z+au,u,v) = E_2(z,u,v)$$

On the other hand, observe that for an integer b

$$\begin{aligned}
E_1(z+bv,u,v) - E_1(z,u,v) &= \lim_{N\to\infty} u^{-1}\left(\sum_{\nu=-N}^{N} \varepsilon_1(\xi+(b+\nu)\tau) + \sum_{\nu=-N}^{N} \varepsilon_1(\xi+\nu\tau)\right)\\
&= \lim_{N\to\infty} u^{-1}\left(\sum_{\nu=-N+b}^{N+b} \varepsilon_1(\xi+\nu\tau) - \sum_{\nu=N}^{-N} \varepsilon_1(\xi+\nu\tau)\right)\\
&= \lim_{N\to\infty} u^{-1}\left(\sum_{\nu=N+1}^{N+b} \varepsilon_1(\xi+\nu\tau) - \sum_{\nu=-N}^{-N-1+b} \varepsilon_1(\xi+\nu\tau)\right)
\end{aligned}$$

By (2.13) $\varepsilon_1(\xi+\nu\tau) \xrightarrow[\nu\to+\infty]{} -\pi i$ and $\varepsilon_1(\xi+\nu\tau) \xrightarrow[\nu\to-\infty]{} \pi i$. Hence one has for integers a, b

$$E_1(z+au+bv,u,v) - E_1(z,u,v) = -\frac{2\pi i b}{u} \tag{2.19}$$

and thus, by (2.18), $A(u,v) = 0$ and $B(u,v) = -\frac{2\pi i b}{u}$.

More generally, for $\Gamma' = u'\mathbb{Z} \oplus v'\mathbb{Z}$ a sub-lattice of Γ, R a set of representatives for $\Gamma/_{\Gamma'}$ containing 0. Then the summation process becomes

$$\sum_{r\in R}\left(\sum_{\mu}{}_{e}\sum_{\nu}{}_{e} E_n(z+\mu u'+\nu v')\right) = \sum_{r\in R}\left(\sum_{\mu}{}_{e}\sum_{\nu}{}_{e} E_n(z+\mu u+\nu v)\right)$$

and for $n \geq 3$

$$E_n'(z,u',v') = \sum_{r\in R} E_n(z+r,u',v') = E_n(z,u,v) \tag{2.20}$$

One can show that in this case

$$E_1'(z,u',v') = \sum_{r\in R} E_1(z+r,u',v') = E_1(z,u,v) + \frac{2\pi i C_r}{uu'}z - \frac{\pi i\overline{\nu}}{u'} \tag{2.21}$$

and by differentiating

$$E_2'(z,u',v') = \sum_{r\in R} E_2(z+r,u',v') = E_2(z,u,v) + \frac{2\pi i C_r}{uu'}$$

where C_r is a constant that depends on $r \in R$ and $\overline{\mu}u' + \overline{\nu}v' = 2\sum_{r\in R} r$.

Remark 2.2.6. In particular, E_n is periodic with respect to Γ for all $n \geq 2$. We will later see (by introducing the modified Eisenstein series E_n^* in §2.2.9) that in order to make up for the periodicity of E_1, we will have to sacrifice complex analyticity .

2.2.7. Addition formula for Eisenstein series

Back to our previous task of deriving equations of the E_n functions, recall the slightly more developed addition formula (2.8) for trigonometric functions, expressed in the notations of §2.2.1 for $\xi + \nu\tau$, $\xi' + \nu'\tau = \xi' + (\rho - \nu)\tau$:

$$\begin{aligned} 2\epsilon_3(\xi + \xi' + \rho\tau)(\varepsilon_1(\xi + \nu\tau) + \varepsilon_1(\xi' + (\rho - \nu)\tau)) = {} & \epsilon_2(\xi + \nu\tau)\epsilon_2(\xi' + (\rho - \nu)\tau) \\ & - \epsilon_1(\xi + \nu\tau)\epsilon_2(\xi + \xi' + \rho\tau) \\ & - \epsilon_1(\xi' + (\rho - \nu)\tau)\epsilon_2(\xi + \xi' + \rho\tau) \end{aligned}$$

We have shown that the series in (2.12) is absolutely convergent for $n \geq 2$. Hence, by applying Eisenstein's summation process to both sides, one gets

$$\begin{aligned} 2\epsilon_3(\xi + \xi' + \rho\tau) \sum_{\nu}{}_e(\varepsilon_1(\xi + \nu\tau) + \varepsilon_1(\xi' + (\rho - \nu)\tau)) = {} & \sum_{\nu} \epsilon_2(\xi + \nu\tau)\epsilon_2(\xi' + (\rho - \nu)\tau) \\ & - \epsilon_2(\xi + \xi' + \rho\tau) \sum_{\nu} \epsilon_1(\xi + \nu\tau) \\ & - \epsilon_2(\xi + \xi' + \rho\tau) \sum_{\nu} \epsilon_1(\xi' + (\rho - \nu)\tau) \end{aligned}$$

Now, by summing over ρ (note that again by absolute convergence, the summation over ν and $\rho - \nu$ can be performed independently, and we may as well interchange it with a summation over ρ and $\rho - \nu$ respectively) one gets

$$\begin{aligned} \sum_{\rho} 2u\epsilon_3(\xi + \xi' + \rho\tau)\left[E_1(z, u, v) + E_1(z' + \rho\nu, u, v)\right] = {} & \sum_{\rho-\nu}\sum_{\nu} \epsilon_2(\xi + \nu\tau)\epsilon_2(\xi' + (\rho - \nu)\tau) \\ & - \sum_{\rho-\nu}\sum_{\rho} \epsilon_2(\xi + \xi' + \rho\tau)\epsilon_1(\xi + (\rho - \nu)\tau) \\ & - \sum_{\rho-\nu}\sum_{\rho} \epsilon_2(\xi + \xi' + \rho\tau)\epsilon_1(\xi' + (\rho - \nu)\tau) \\ = {} & u^4 E_2(z, u, v)E_2(z', u, v) \\ & - u^4 E_2(z, u, v)E_2(z + z', u, v) \\ & - u^4 E_2(z', u, v)E_2(z + z', u, v) \end{aligned}$$

Since $E_1(z' + \rho v, u, v) = E_1(z', u, v) - \frac{2\pi i}{u}\rho$, the RHS above becomes

$$2u^4 E_3(z + z')\left[E_1(z, u, v) + E_1(z', u, v)\right] - 2\pi i \sum_{\rho} 2\rho\, \epsilon_3(\xi + \xi' + \rho\tau)$$

and as $-2\rho\, \epsilon_3(\xi + \xi' + \rho\tau) = \frac{d}{d\tau}\epsilon_2(\xi + \xi' + \rho\tau)$, one has

$$2u^4 E_3(z + z')\left[E_1(z, u, v) + E_1(z', u, v)\right] - 2\pi i \frac{d}{d\tau} \sum_{\rho} \epsilon_2(\xi + \xi' + \rho\tau)$$

Finally one obtains the analogous addition formula of (2.8) given by

$$\begin{aligned} & 2E_3(z + z')\left[E_1(z, u, v) + E_1(z', u, v)\right] - \frac{2\pi i}{u}\frac{d}{dv}E_2(z + z', u, v) \\ & \quad = E_2(z, u, v)E_2(z', u, v) - E_2(z, u, v)E_2(z + z', u, v) - E_2(z', u, v)E_2(z + z', u, v) \end{aligned} \tag{2.22}$$

We now fix $z \in \mathbb{C} \setminus \Gamma$ and regard both sides of the equation above as meromorphic functions of z'. By (2.16) we have the power series expansions

$$E_2(z',u,v) = \frac{1}{z'^2} + \sum_{k=1}^{\infty}(2k-1)e_{2k}z'^{2k-2} = \frac{1}{z'^2} - e_2 + 3e_4z'^2 + \dots$$
$$E_3(z',u,v) = \frac{1}{z'^3} - \sum_{k=1}^{\infty}\binom{2k-1}{2}e_{2k}z'^{2k-3} = \frac{1}{z'^3} - 3e_4z - 10e_6z'^3 + \dots$$

On the other hand, for all $n \leq 1$, the Taylor series of E_n around $z' = 0$ is given by

$$E_n(z+z',u,v) = \sum_{k=0}^{\infty}\frac{E^{(k)}(z)}{k!}z'^k$$

Hence we have

$$E_2(z+z',u,v) = E_2(z,u,v) - 2E_3(z)z' + 3E_4(z)z'^2 - \dots$$
$$E_3(z+z',u,v) = E_3(z) - 3E_4(z)z' + 6E_5(z)z'^2 + \dots$$

Developing both sides of (2.22) into power series around $z' = 0$ gives something like

$$\begin{aligned} 2\left(E_3(z) - 3E_4(z)z' + \dots\right)&\left(\frac{1}{z'} - e_2z' + \dots\right) + 2E_3(z)E_1(z,u,v) + \dots \\ &= -\left(\frac{1}{z'^2} + e_2 + \dots\right)\left(-2E_3(z)z' + 3E_4(z)z'^2 + \dots\right) - E_2^2(z,u,v) + \dots \end{aligned}$$

since the terms $E_2(z,u,v)E_2(z',u,v)$ cancel. This finally implies, by equality of coefficients, the following identity

$$2E_3(z) - 6E_4(z) + 2E_3(z)E_1(z,u,v)\frac{2\pi i}{u}\frac{d}{dv}E_2(z,u,v) = 2E_3(z) - 3E_4(z) - 2E_2^2(z,u,v)$$
$$\Leftrightarrow \quad \frac{2\pi i}{u}\frac{d}{dv}E_2(z,u,v) = 3E_4(z) - 2E_3(z)E_1(z,u,v) - 2E_2^2(z,u,v). \tag{2.23}$$

By integrating in both sides of (2.23) and remarking that

$$\begin{aligned} \frac{d}{dz}\left[E_1(z,u,v)E_2(z,u,v)\right] &= \left(\frac{d}{dz}E_1(z,u,v)\right)E_2(z,u,v) + E_1(z,u,v)\left(\frac{d}{dz}E_2(z,u,v)\right) \\ &= 2E_3(z)E_1(z,u,v) + 2E_2^2(z,u,v) \end{aligned}$$

one gets the analogous of (2.6), namely

$$E_3(z) = E_2(z,u,v)E_1(z,u,v) + \frac{2\pi i}{u}\frac{d}{dv}E_1(z,u,v) \tag{2.24}$$

Remark 2.2.8 (Weierstrass invariant). Note that by the same reasoning above on (2.22) by considering both sides as functions of $z+z'$ and developing at $z+z' = 0$ we get

$$\begin{aligned} \frac{2\pi i}{u}\frac{d}{dv}e_2(z,u,v) &= -E_4(z) - 2e_2E_2(z,u,v) - 2E_2^2(z,u,v) = 5e_4 - e_2^2 \\ \Leftrightarrow \quad E_4(z) &= (E_2(z,u,v) - e_2)^2 - 5e_4 \\ \Leftrightarrow \quad 6E_4(z) &= \frac{d^2}{dz^2}E_2(z,u,v) = 6(E_2(z,u,v) - e_2)^2 - 30e_4 \end{aligned} \tag{2.25}$$

Integrating this formula gives

$$\begin{aligned} & E_3^2(z) = (E_2(z,u,v) - e_2)^3 - 15e_4(E_2(z,u,v) - e_2) - 10\left(\frac{\pi i}{2u}\frac{d}{dv}e_4 - e_2e_4\right) \\ \Leftrightarrow \qquad & 4E_3^2(z) = 4(E_2(z,u,v) - e_2)^3 - 60e_4(E_2(z,u,v) - e_2) - 140e_6 \end{aligned}$$

This give the known identities

$$\begin{aligned} \wp'' &= 6\wp^2 - \frac{1}{2}g_2 \\ \wp'^2 &= 4\wp^3 - g_2\wp - g_3 \end{aligned}$$

The constants $g_2 = 60e_4$ and $g_3 = 140e_6$ are known as the *Weierstrass invariant.*

2.2.9. Modified Eisenstein series E_n^*

We have seen in the end of §2.2.5 that E_n is periodic with respect to Γ for all $n \geq 2$. We make up for the periodicity of E_1 as follows: Let u, v be generators of Γ as above, and recall $\tau = \Im(\frac{v}{u}) > 0$. One sees that

$$u\overline{v} - v\overline{u} = u\overline{u}\left(\frac{\overline{v}}{\overline{u}} - \frac{v}{u}\right) = -|u|^2(2i\tau) = -2\pi iA \tag{2.26}$$

where $A = \frac{|u|^2}{\pi}\tau > 0$. Now for all $z \in \mathbb{C}$ we can write $z = \alpha u + \beta v$ where $\alpha, \beta \in \mathbb{R}$. Observe that

$$z\overline{u} - u\overline{z} = \beta(v\overline{u} - u\overline{v}) \Rightarrow \beta(z) := \beta = \frac{z\overline{u} - u\overline{z}}{2\pi iA}$$

We pose

$$\begin{aligned} E_1^*(z,\Gamma) &:= E_1(z,u,v) + \frac{2\pi i}{u}\beta(z) \\ E_2^*(z,\Gamma) &:= -\frac{\partial}{\partial z}E_1^*(z,\Gamma) = E_2(z,u,v) + \frac{\overline{u}}{Au} \end{aligned} \tag{2.27}$$

Remark 2.2.10. One clearly sees that E_1^* is odd and periodic with respect to Γ. Moreover, it depends only on the lattice Γ and not on the choice of the generators: Indeed, the formula (2.21) in the end of §2.2.5 shows that any change in the generators of Γ produces a function that differs from $E_1(z,u,v)$ only by a linear factor $Uz + V$. Thus E_1^* can be characterised as the unique Γ-periodic function that differs from all the $E_1'(z,u',v')$ by a real linear factor.

Define the following differential operators

$$\begin{aligned} \mathscr{D} &:= z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \\ \overline{\mathscr{D}} &:= \overline{z}\frac{\partial}{\partial z} + \overline{u}\frac{\partial}{\partial u} + \overline{v}\frac{\partial}{\partial v} \end{aligned}$$

Clearly, $\overline{\mathscr{D}}$ is invariant under any real linear substitution in z, u or in v and it commutes with $\frac{\partial}{\partial z}$. We define for integers $b > a \geq 0$

$$E_{a,b}(z,\Gamma) = \sum_{w\in\Gamma}^{e} \frac{(\overline{z}+\overline{w})^a}{(z+w)^b}$$

Now, for $b - a \geq 3$ this series is absolutely and locally uniformly convergent. Moreover, one sees that

$$\overline{\mathscr{D}}E_1(z,\Gamma) = \sum_{\mu,\nu}^{e} -\frac{\overline{z}}{(z+\mu u+\nu v)^2} - \frac{\mu\overline{u}}{(z+\mu u+\nu v)^2} - \frac{\nu\overline{v}}{(z+\mu u+\nu v)^2} = -\sum_{w\in\Gamma}^{e} \frac{\overline{z}+\overline{w}}{(z+w)^2}$$

Similarly,

$$\overline{\mathscr{D}}^{(a)}E_1(z,\Gamma) = (-1)^a a! \sum_{w\in\Gamma}^{e} \frac{(\overline{z}+\overline{w})^a}{(z+w)^{a+1}}$$

and take the partial derivatives with respect to z

$$\begin{aligned}
\left(\frac{\partial}{\partial z}\right)^{b-a-1} \overline{\mathscr{D}}^{(a)}E_1(z,\Gamma) &= (-1)^a a! \sum_{w\in\Gamma}^{e} (\overline{z}+\overline{w})^a \left(\frac{\partial}{\partial z}\right)^{b-a-1} \left(\frac{1}{(z+w)^{a+1}}\right) \\
&= (-1)^a a! \sum_{w\in\Gamma}^{e} (\overline{z}+\overline{w})^a \left(\frac{\partial}{\partial z}\right)^{b-a-1} E_{a+1}(z,\Gamma) \\
&= (-1)^a a! \sum_{w\in\Gamma}^{e} (\overline{z}+\overline{w})^a (-1)^{b-a-1}(a+1)(a+2)\dots(b-1)E_b(z,\Gamma) \\
&= (-1)^{b-1}(b-1)! \sum_{w\in\Gamma}^{e} \frac{(\overline{z}+\overline{w})^a}{(z+w)^b}
\end{aligned}$$

Hence

$$\begin{aligned}
E_{a,b}(z,\Gamma) &= \sum_{w\in\Gamma}^{e} \frac{(\overline{z}+\overline{w})^a}{(z+w)^b} = \frac{(-1)^{b-1}}{(b-1)!}\overline{\mathscr{D}}^{(a)} \left(\frac{\partial}{\partial z}\right)^{b-a-1} E_1(z,\Gamma) \\
e_{a,b} &= \frac{(-1)^a}{(b-1)\dots(b-a)}\overline{\mathscr{D}}^{(a)}(e^*_{b-a})
\end{aligned}$$

where $e_{a,b}$ is the values of $E_{a,b}(z) - \frac{\overline{z}^a}{z^b}$ near $z = 0$.

Similarly, define

$$\begin{aligned}
E^*_{a,b}(z,\Gamma) &= \frac{(-1)^{b-1}}{(b-1)!}\overline{\mathscr{D}}^{(a)} \left(\frac{\partial}{\partial z}\right)^{b-a-1} E^*_1(z,\Gamma) \\
e^*_{a,b} &= \frac{(-1)^a}{(b-1)\dots(b-a)}\overline{\mathscr{D}}^{(a)}(e^*_{b-a})
\end{aligned} \tag{2.28}$$

Remark 2.2.11. Note that

- $E_{a,b} = E^*_{a,b}$ whenever $b - a \geq 3$, as E^*_2 differs from E_2 by a constant.

- For all $n \geq 0$, $E_{0,n} = E_n$ and $E^*_{0,n} = E^*_n$.

Observe that since $E_n(z,u,v)$ is homogeneous of degree $-n$, one has for all t

$$E_n(tz,tu,tv) = t^{-n}E_n(z,u,v) \tag{2.29}$$

Differentiating (2.29) with respect to t gives

$$\frac{\partial E_n}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial E_n}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial E_n}{\partial v}\frac{\partial v}{\partial t} = \left(z\frac{\partial E_n}{\partial z} + u\frac{\partial E_n}{\partial u} + v\frac{\partial E_n}{\partial v}\right)(tz,tu,tv) = -nt^{-n-1}E_n(z,u,v)$$

Taking $t = 1$ gives

$$\mathscr{D}(E_n) = z\frac{\partial E_n}{\partial z} + u\frac{\partial E_n}{\partial u} + v\frac{\partial E_n}{\partial v} = -nE_n(z,u,v)$$

In particular, as E_1 is homogeneous of degree -1 in z,u,v and of degree 0 in $\overline{z},\overline{u},\overline{v}$

$$\mathscr{D}(E_1)(z) = -E_1(z) \tag{2.30}$$

Hence, one sees that

$$\begin{aligned}
E^*_{1,2} &= -\overline{\mathscr{D}}(E^*_1) = -\overline{\mathscr{D}}(E_1) + \frac{\overline{u}}{u}\frac{z\overline{u}-\overline{z}u}{Au} = -\overline{\mathscr{D}}(E_1) + A(E_2 - E^*_2)(E^*_1 - E_1) \\
&= -\overline{\mathscr{D}}(E_1) + A(E^*_2E_1 - E^*_1E_2) + \frac{2\pi i A}{u}\frac{\partial}{\partial v}E_1 + A(E^*_3 - E^*_2E^*_1) && \text{by (2.24)} \\
&= -\overline{\mathscr{D}}(E_1) + \frac{z\overline{u}-\overline{z}u}{u}\frac{\partial}{\partial z}E_1 - \frac{\overline{u}}{u}E_1 + \frac{u\overline{v}-v\overline{u}}{u}\frac{\partial}{\partial v}E_1 + A(E^*_3 - E^*_2E^*_1) && \text{by (2.26)}
\end{aligned}$$

Now, using (2.30), the first three terms cancel out and one finally gets

$$E^*_{1,2} = A(E^*_3 - E^*_2E^*_1) \tag{2.31}$$

By taking $n-1$ successive derivations, the above equation becomes

$$E^*_{1,n+1} = -\frac{1}{n}\overline{\mathscr{D}}(E^*_1) = \frac{A}{2}\left(nE^*_{n+2} - E^*_1E^*_{n-1} - \cdots - E^*_{n+1}E^*_1 + E^*_{n+2}\right) \tag{2.32}$$

More generally, for $b > a \geq 0$ one has the formula

$$E^*_{a,b} = \frac{A^a}{2^a(b-1)\dots(b-a)}P_{a,b}\left(E^*_1,\dots,E_{a+b}\right)$$

where $P_{a,b} \in \mathbb{Q}[E^*_1,\dots,E^*_{a+b}]$ is a polynomial of degree $a+1$.

In the same fashion, performing almost the same calculations as the ones that lead formula (2.32) one gets

$$\begin{aligned}
e^*_{1,2k+1} &= -\frac{1}{2k}\overline{\mathscr{D}}(e^*_{2k}) = \frac{A}{2}\left((2k+1)e^*_{2k+2} - e^*_2e^*_{2k} - \cdots - e^*_{2k}E^*_2 + e^*_{2n+2}\right) \\
e^*_{a,b} &= \frac{A^a}{2^a(b-1)\dots(b-a)}Q_{a,b}\left(e^*_2,\dots,e_{a+b}\right)
\end{aligned} \tag{2.33}$$

where $Q_{a,b} \in \mathbb{Q}[e^*_1,\dots,e_{a+b}]$ is a polynomial of degree $a+1$. Note that $Q_{a,b} = 0$ for all odd values of $a+b$.

2.3. The Eisenstein-Kronecker series

Let $\Gamma = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} \subset \mathbb{C}$ be a lattice as in §2.2, and

$$A(\Gamma) = \frac{(\text{Area of } \Gamma)}{\pi} = \frac{1}{\pi}\Im\left(\frac{v}{u}\right) = \frac{\omega_2\overline{\omega_1} - \omega_1\overline{\omega_2}}{2\pi i}.$$

We define for $z, z' \in \mathbb{C}$

$$\langle\, z, z' \,\rangle_\Gamma := \exp\left(\frac{z\overline{z'} - z'\overline{z}}{A(\Gamma)}\right).$$

Note that, by direct calculations, one easily verifies that $\langle\, \cdot, \cdot \,\rangle_\Gamma$ defines a complex pairing since:

(i) $\langle\, z, z' \,\rangle_\Gamma = \langle\, -z', z \,\rangle_\Gamma = \langle\, z', z \,\rangle_\Gamma^{-1}$

(ii) $\langle\, az, z' \,\rangle_\Gamma = \langle\, z, \overline{a}z' \,\rangle_\Gamma$ for any $a \in \mathbb{C}$

(iii) $z \in \Gamma \Leftrightarrow \langle\, z, \gamma \,\rangle_\Gamma = 1$ for any $\gamma \in \Gamma$.

2.3.1. Eisenstein-Kronecker-Lerch series

Definition 2.3.2 (Eisenstein-Kronecker-Lerch series)**.** Le a be a positive integer and $z_0, z_0' \in \mathbb{C}$ be complex numbers. we define the Eisenstein-Kronecker-Lerch series to be the holomorphic functions on the domain $D(s) = \{s \in \mathbb{C} \mid \Re(s) > 1 + \frac{a}{2}\}$ defined by

$$K_a^*(z_0, z_0', s; \Gamma) := \sum_{\gamma\in\Gamma}^{*} \frac{(\overline{z_0} + \overline{\gamma})^a}{|z_0 + \gamma|^{2s}} \langle\, \gamma, z_0' \,\rangle_\Gamma \tag{2.34}$$

Where the sum $\overset{*}{\sum}$ goes over all $\gamma \in \Gamma$ except $-z_0$ when $z_0 \in \Gamma$.

This series converges absolutely for $\Re(s) > 1 + \frac{a}{2}$. Indeed, for $\epsilon > 0$ and for all $\gamma \in \Gamma \smallsetminus \mathbb{C} \cap B\left(0, \max\left\{\frac{|z_0|}{\epsilon}, \frac{|z_0'|}{2}\right\}\right)$:

$$\begin{cases} |z_0 + \gamma|^a & \le & (|z_0| + |\gamma|)^a & \le & (1+\epsilon)^a |\gamma|^a \\ \left|\frac{1}{|\overline{z_0}+\overline{\gamma}|^{2s}}\right| & \le & \left|\frac{1}{|\overline{z_0}|-|\overline{\gamma}|^{2s}}\right| & \le & \frac{1}{(1-\epsilon)^{2\Re(s)}|\gamma|^{2\Re(s)}} \end{cases}$$

On the other hand, observe that

$$\frac{\gamma\overline{z_0'} + \overline{\gamma}z_0'}{A} \in i\,\mathbb{R} \quad \Rightarrow \quad \left|\langle\, \gamma, z_0' \,\rangle_\Gamma\right| = 1$$

Hence, by letting $\epsilon \longrightarrow 0$ one finally sees that

$$\left|\frac{(\overline{z_0} + \overline{\gamma})^a}{|z_0 + \gamma|^{2s}} \langle\, \gamma, z_0' \,\rangle_\Gamma\right| \le \frac{2}{A}\sum_{k=0}^{\infty} |\gamma|^{-(2\Re(s)-a)}$$

Which, by the Convergence Lemma (A.1.1), converges to a holomorphic function for all values $\mathfrak{R}(s) > \frac{a}{2} + 1$.

By abuse of notations, we will often omit "Γ" when there is no fear of confusion. For fixed complex numbers $z_0, z_0' \in \mathbb{C}$, one can extend the domain of the Eisenstein-Kronecker-Lerch series as follows:

Proposition 2.3.3. *Let a be a positive integer, $z_0, z_0' \in \mathbb{C}$ be complex numbers.*

(i) The function $K_a^*(z_0, z_0', s)$ admits an analytic continuation into a meromorphic function on $\mathbb{C}$ with possible poles only in the following cases:

(a) $a = 0$ and $z \in \Gamma$: In this case, the function $K_a^*(z_0, z_0', s)$ has a simple pole at $s = 0$ with residue $-\langle z_0', z_0 \rangle$.

(b) $a = 0$ and $z_0 \in \Gamma$: In this case, the function $K_a^*(z_0, z_0', s)$ has a simple pole at $s = 1$ with residue A^{-1}.

Moreover, for $a \geq 1$ the function

$$K_a^*(z_0, z_0', s) = \frac{1}{\Gamma(s)} \left(I_{A^{-1}}(a, z_0, z_0', s) + A^{a+1-2s} I_{A^{-1}}(a+1, z_0', z_0, 1-s) \langle z_0', z_0 \rangle_\Gamma \right) \tag{2.35}$$

is analytic in (z_0, z_0') for all $\gamma \in \Gamma \smallsetminus \{-z_0\}$ with

$$I_{A^{-1}}(a, z_0, z_0', s) = \int_{A^{-1}}^{\infty} \sum_{\gamma \in \Gamma} \exp\left(-t|z_0 + \gamma|^2\right) (\overline{z_0} + \overline{\gamma})^a \langle \gamma, z_0' \rangle_\Gamma t^{s-1}\, dt$$

(ii) The function $K_a^*(z_0, z_0', s)$ satisfies the functional equation

$$K_a^*(z_0, z_0', s) = A^{a+1-2s} \frac{\Gamma(a+1-s)}{\Gamma(s)} K_a^*(z_0', z_0, a+1-s) \langle z_0', z_0 \rangle_\Gamma \tag{2.36}$$

Proof. (i) Fix z_0, z_0' in $\mathbb{C}$ and consider the Mellin transform of $K_a^*(z_0, z_0', s; \Gamma)$ as a function of z_0. Namely, the function

$$\Gamma(s) K_a^*(z_0, z_0', s) := \int_0^{\infty} \sideset{}{^*}\sum_{\gamma \in \Gamma} \exp\left(-t|z_0 + \gamma|^2\right) (\overline{z_0} + \overline{\gamma})^a \langle \gamma, z_0' \rangle t^{s-1}\, dt \tag{2.37}$$

Pose

$$\theta_{a,t}(z, z_0') := \sum_{\gamma \in \Gamma} \exp\left(-t|z + \gamma|^2\right) (\overline{z} + \overline{\gamma})^a \langle \gamma, z_0' \rangle \tag{2.38}$$

$$\theta_{a,t}^*(z, z_0') := \sideset{}{^*}\sum_{\gamma \in \Gamma} \exp\left(-t|z + \gamma|^2\right) (\overline{z} + \overline{\gamma})^a \langle \gamma, z_0' \rangle \tag{2.39}$$

One has the following transformation formula:

Lemma 2.3.4. *The above function satisfies the following functional equation*

$$\theta_{a,t}(z_0, z_0') = \frac{\langle z_0', z_0 \rangle}{(At)^{a+1}} \theta_{a,A^{-2}t^{-1}}(z_0', z_0)$$

Proof. See appendix A, lemma A.1.4. □

Note that

$$\begin{cases} \theta^*_{a,t}(z_0, z_0') = \theta_{a,t}(z_0, z_0') - \langle z_0', z_0 \rangle & \text{If } a = 0 \text{ and } z_0 \in \Gamma; \\ \\ \theta^*_{a,t}, (z_0, z_0') = \theta_{a,t}(z_0, z_0') & \text{Otherwise.} \end{cases} \tag{2.40}$$

Let $T > 0$, we decompose the integral (2.37) into

$$\int_0^T \theta^*_{a,t}(z, z_0') t^{s-1}\, dt + \int_T^\infty \theta^*_{a,t}(z, z_0') t^{s-1}\, dt =: I_0(t, z_0', z_0, s-1) + I_\infty(t, z_0', z_0, s-1)$$

For $\epsilon > 0$ and for all $\frac{|z|}{2} \leq |\gamma| \leq N$ and $t \geq T > 0$

$$\begin{aligned} & -t|\gamma + z|^2 \leq -t\big||\gamma| - |z|\big|^2 \leq -t(1-\epsilon)^2|\gamma|^2 \\ \Rightarrow \quad & |\overline{\gamma} + \overline{z}| e^{-t|\gamma+z|^2} \leq (1+\epsilon)^a |\gamma|^a e^{-t(1-\epsilon)^2|\gamma|^2} \end{aligned}$$

Hence, by letting $\epsilon \longrightarrow 0$ and making the change of variable $u = |\gamma|^2 t$

$$\begin{aligned} |I_\infty| &\leq \sum_\gamma |\gamma|^a \int_T^\infty e^{-|\gamma|^2 t} t^{\Re(s)-1}\, dt \\ &\leq \sum_\gamma \frac{1}{|\gamma|^{2\Re(s)-a}} \int_T^\infty e^{-u} u^{\Re(s)-1}\, du < \infty \end{aligned}$$

where $\Gamma(s,T) = \int_T^\infty e^{-u} u^{s-1}\, du$ denotes the "upper" incomplete Γ-function, and converges for all real values of s. Hence I_∞ is absolutely and uniformly convergent inside compact sets for all values of s.

Now back to $I_T(t, z_0', z_0, s-1)$, one distinguishes between the following cases:

- If $a = 0$ and $z_0 \in \Gamma$: Make the change of variable $t = A^{-2}u^{-1}$

$$\begin{aligned}\int_0^T \theta^*_{0,t}(z_0, z_0')t^{s-1}\,dt &= \int_0^T \theta_{0,t}(z, z_0')t^{s-1}\,dt - \int_0^T \langle\, z_0', z_0\,\rangle t^{s-1}\,dt\\ &= \langle\, z_0', z_0\,\rangle \int_0^T (At)^{-1}\theta_{0,A^{-2}t^{-1}}(z_0', z_0)\,dt - \langle\, z_0', z_0\,\rangle \frac{T^s}{s}\\ &= \langle\, z_0', z_0\,\rangle \int_T^\infty A^{1-2s}\theta_{0,u}(z_0', z_0)u^{1-s}\,du - \langle\, z_0', z_0\,\rangle \frac{T^s}{s}\\ &= \langle\, z_0', z_0\,\rangle \left(I_\infty(u, z_0, z_0', 1-s, 0) - \frac{T^s}{s}\right)\end{aligned}$$

- If $a = 0$ and $z_0' \in \Gamma$:

$$\begin{aligned}\int_0^T \theta^*_{0,t}(z_0, z_0')t^{s-1}\,dt &= \langle\, z_0', z_0\,\rangle \int_0^T (At)^{-1}\theta^*_{0,A^{-2}t^{-1}}(z_0', z_0)\,dt\\ &= \langle\, z_0', z_0\,\rangle \int_0^T (At)^{-1}\theta_{0,t}(z, z_0')t^{s-1}\,dt - \langle\, z_0', z_0\,\rangle \int_0^T \langle\, z_0, z_0'\,\rangle (At)^{-1}t^{s-1}\,dt\\ &= \langle\, z_0', z_0\,\rangle \int_T^\infty A^{1-2s}\theta_{0,u}(z_0', z_0)u^{1-s}\,du - A^{-1}\frac{T^{s-1}}{s-1}\\ &= \langle\, z_0', z_0\,\rangle I_\infty(u, z_0, z_0', 1-s, 0) - A^{-1}\frac{T^{s-1}}{s-1}\end{aligned}$$

- For all $a > 0$:

$$\begin{aligned}\int_0^T \theta^*_{a,t},(z, z_0')t^{s-1}\,dt = \int_0^T \theta_{a,t}(z, z_0')t^{s-1}\,dt &= \langle\, z_0', z_0\,\rangle \int_0^T (At)^{-1}\theta_{a,A^{-2}t^{-1}}(z_0', z_0)\,dt\\ &= \langle\, z_0', z_0\,\rangle \int_T^\infty A^{a+1-2s}\theta_{a,u}(z_0', z_0)u^{a+1-s}\,du\\ &= \langle\, z_0', z_0\,\rangle I_\infty(u, z_0, z_0', a+1-s, 0)\end{aligned}$$

Finally, the analytic continuation of the Eisenstein-Kronecker-Lerch series is given by:

- If $a = 0$ and $z_0 \in \Gamma$:

$$\begin{aligned}\Gamma(s)K^*_a(z_0, z_0', s) = I_\infty(t, z_0', z_0, s-1, 0) &+ \langle\, z_0', z_0\,\rangle I_\infty(t, z_0, z_0', 1-s, 0)\\ &- \langle\, z_0', z_0\,\rangle \frac{T^s}{s}\end{aligned}$$

  with a simple pole at $s = 0$ and $\operatorname{res}_{s=0}(K^*_a, s) = -\langle\, z_0', z_0\,\rangle$

- If $a = 0$ and $z_0' \in \Gamma$:

$$\Gamma(s)K_a^*(z_0, z_0', s) = I_\infty(t, z_0', z_0, s-1, 0) + \langle z_0', z_0 \rangle I_\infty(t, z_0, z_0', 1-s, 0) \\ - A^{-1}\frac{T^{s-1}}{s-1}$$

  with a simple pole at $s = 1$ and $\mathrm{res}_{s=1}(K_a^*, s) = -A^{-1}$

- For all $a > 0$

$$\Gamma(s)K_a^*(z_0, z_0', s) = I_\infty(t, z_0', z_0, s-1, 0) + \langle z_0', z_0 \rangle I_\infty(t, z_0, z_0', a+1-s, 0) \quad (2.41)$$

Taking $T = A^{-1}$, using lemma A.1.4 and making again the change of variable $u = A^{-2}t^{-1}$ in $I_\infty(t, z_0', z_0, s-1) = I_{A^{-1}}(u, z_0, z_0', a+1-s, 0)$, one gets the functional equation in (2.36).

□

Remark 2.3.5. The parity of the function $K_a^*(z_0, z_0', s)$ depends on the parity of a: Indeed, it is clear from 2.34 for $\Re(s) > 1 + \frac{a}{2}$, and by analytic continuation, to all values of s. Moreover, it is Γ-periodic.

2.3.6. Eisenstein-Kronecker numbers

One wants to relate the Eisenstein-Kronecker-Lerch series for integers a, b as in the case of the modified Eisenstein series in §2.2.9. To do so, consider for all $a \geq 0$, $z \in \mathbb{C} \smallsetminus \Gamma$ and $\omega \in \mathbb{C}$ the function

$$K_a(z, \omega, s) := \sum_{\gamma \in \Gamma} \frac{(\overline{z} + \overline{\gamma})^a}{|z+\gamma|^{2s}} \langle \gamma, w \rangle \quad (2.42)$$

Then one sees that in this case, one has $K_a(z, \omega, s) = K_a^*(z, \omega, s)$. We will write K_a for all $z \in \mathbb{C} \smallsetminus \Gamma$ and we always fix $z_0, w_o \in \mathbb{C}$ for K_a^*.

For $\Re(s) > 1 + \frac{a}{2}$ and $a > 0$, direct calculations show that

$$\begin{aligned} \frac{\partial}{\partial z} K_a(z, \omega, s) &= -sK_{a+1}(z, \omega, s+1) \\ \frac{\partial}{\partial \overline{z}} K_a(z, \omega, s) &= (a-s)K_{a-1}(z, \omega, s) \end{aligned} \quad (2.43)$$

One also easily deduces the following identity:

$$\overline{\mathscr{D}} K_a(z, 0, s) = -sK_{a+2}(z, 0, s+1) \quad (2.44)$$

More generally, and by analytic continuations, one has for all values of s:

Lemma 2.3.7. *Let $a > 0$, $z \in \mathbb{C} \smallsetminus \Gamma$ and $\omega \in \mathbb{C}$. Then, for all $s \in \mathbb{C}$, the function $K_a^*(z, \omega, s)$ satisfies the following differential equations*

- $\frac{\partial}{\partial z}K_a(z,\omega,s) = -sK_{a+1}(z,\omega,s+1)$
- $\frac{\partial}{\partial \overline{z}}K_a(z,\omega,s) = (a-s)K_{a-1}(z,\omega,s)$
- $\frac{\partial}{\partial \omega}K_a(z,\omega,s) = -A^{-1}(K_{a+1}(z,\omega,s) - \overline{z}K_a(z,\omega,s))$
- $\frac{\partial}{\partial \overline{\omega}}K_a(z,\omega,s) = A^{-1}(K_{a-1}(z,\omega,s-1) - zK_a(z,\omega,s))$

This shows that $K_a(z,\omega,s)$ is holomorphic in z for all $a \geq 1$, since $K_{a-1}(z,\omega,s)$ has a simple pole at $s=1$ (and only when $a=1$ and $w \in \Gamma$). In this case one has

$$\frac{\partial}{\partial \overline{z}}K_1(z,0,1) = -\operatorname{res}_{s=1}(K_0,s) = -A^{-1}$$

Observe that

$$\frac{\partial}{\partial z_0}K_2^*(z_0,0,2) = -2\sum_{\gamma\in\Gamma}^{*}\frac{(\overline{z_0}+\overline{\gamma})^2}{|z_0+\gamma|^6} = -2\sum_{\gamma}{}_{e}\frac{(\overline{z_0}+\overline{\gamma})^2}{(z_0+\gamma)^3} = -2E_{3,2}(z_0,\Gamma) = \frac{\partial}{\partial z_0}E_2(z_0,u,v)$$
$$\frac{\partial}{\partial \overline{z_0}}K_2^*(z_0,0,2) = 0 = \frac{\partial}{\partial z_0}E_2(z_0,u,v)$$

Hence,

$$K_2^*(z_0,0,2) = E_2(z_0,u,v) + C$$

Similarly,

$$\frac{\partial}{\partial z_0}\left[K_1^*(z_0,0,1) - E_1(z,u,v)\right] = -K_2^*(z_0,0,2) + E_2(z_0,u,v) = -C$$
$$\frac{\partial}{\partial \overline{z_0}}\left[K_1^*(z_0,0,1) - E_1(z,u,v)\right] = -A^{-1}$$

Hence, $K_1^*(z_0,0,1) = E_1(z_0,u,v) + C'z_0 + C''$ where $C', C'' \in \mathbb{R}$.

As K_1^* is Γ-periodic, remark 2.2.10 implies that

$$K_1^*(z_0,0,1) = E_1^*(z_0,\Gamma)$$

and using (2.44), one gets for all $b > a \geq 0$:

$$K_{a+b}^*(z_0,0,b) = E_{a,b}^*(z_0;\Gamma).$$

Definition 2.3.8 (Eisenstein-Kronecker number)**.** Let $z_0, z_0' \in \mathbb{C}$. For any integers $a \geq 0$ and $b \geq 1$, we define the **Eisenstein-Kronecker numbers** to be

$$e_{a,b}^*(z_0,z_0';\Gamma) \coloneqq K_{a+b}^*(z_0,z_0',b;\Gamma)$$

Note that for $z_0 = z_0' = 0$ one has

$$e_{a,b}^*(0,0;\Gamma) = K_{a+b}^*(0,0,b) = e_{a,b}^*(\Gamma)$$

where $e_{a,b}^*$ in the right hand side are the same as (2.28).

3

L-function associated to algebraic Hecke characters

The L-functions are a very important class of classical functions, that arise in algebraic number theory. One of the most important ones is known as the Riemann ζ-function, the Dedekind ζ-function or the Dirichlet L-series. In this section we introduce another important class of L-function associated to number fields : the Hecke L-function. We then show, in the case of an imaginary quadratic field, how it is -under complex multiplication-related to the Eisenstein-Kronecker numbers $e^*_{a,b}$.

3.1. Hecke characters

In all what follows, K will denote an algebraic number field, and $\mathcal{O}_K$ its ring of integers.

3.1.1. Motivations

Let G be an abelian group. We will call a *character* of G every group homomorphism

$$\chi : G \longrightarrow \mathbb{C}^\times$$

If G is finite, then $\chi(g)$ is a root of unity for all $g \in G$.

Definition 3.1.2 (Dirichlet character)**.** Let $n \geq 0$ be a positive integer, and χ_n a character of $G_n := \left(\mathbb{Z}/_{n\mathbb{Z}}\right)^\times$. The **Dirichlet character** induced by χ_n is the arithmetic function

$$\begin{aligned} \chi : G_n &\longrightarrow \mathbb{Z} \\ k &\longmapsto \chi(k) = \begin{cases} \chi_n(k) & \text{if } (n,k) = 1, \\ 0 & \text{else} \end{cases} \end{aligned}$$

For a given $n \geq 0$, every Dirichlet character satisfies:

$$\chi(k+n) = \chi(k) \quad \text{and} \quad \chi(kk') = \chi(k)\chi(k')$$

for all $k, k' \in \mathbb{Z}$. A Dricihlet character χ is said to be **primitive** if for all $d \mid n$, χ cannot factor through

$$\begin{tikzcd} G_n \arrow[rr, "\chi"] \arrow[dr, "\pi"'] & & \mathbb{C}^\times \\ & G_d \arrow[ur, "\chi'"'] & \end{tikzcd}$$

where χ' is some Dirichlet character on G_d. Such an n is called the **conductor** of χ and will be denoted by $f_\chi := n$. When $f_\chi = 1$, the character χ is said to be **principal**. Finally, if its codomain is $\mathbb{S}^1$, the character is said to be **unitary**. An example of a primitive Dirichlet character on G_p where $p \neq 2$ is prime, is given by the Legendre symbol

$$\chi_p(k) := \left(\frac{k}{p}\right) \equiv k^{\frac{p-1}{2}} \mod p$$

Definition 3.1.3 (Dirichlet L-functions)**.** Let χ be a Dirichlet character modulo n. For a complex number s with $\Re(s) > 1$, the **Dirichlet L-function** associated to the Dirichlet character χ is the formal series

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Note that if χ is not principal, then the series converge absolutely for for all $\Re(s) > 1$ and the function $L(s,\chi)$ has an analytic continuation to the entire complex plane. (When χ is principal, it has a simple pole $s = 1$).

Theorem 3.1.4 (Euler product)**.** *Let χ be a Dirichlet character. Then for all $s \in \mathbb{C}$ such that $\Re(s) > 1$*

$$L(s,\chi) = \prod_{p \text{ prime}} (1 - \chi(p)p^{-s})^{-1}$$

Proof. See for example [Apo76]. □

One sees from this theorem that the Dirichlet L-function is somehow related to $K = \mathbb{Q}$. One would like to construct a more general L-function for an arbitrary number fields K. By analogy to the construction above, one would think of constructing a characters on $G = \left(\mathcal{O}_K/\mathfrak{a}\right)^\times$ where $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$. The problem though is that for such an ideal $\mathfrak{a}$; the inverse of the coset $x + \mathfrak{a}$ is in general not $x^{-1} + \mathfrak{a}$ since x^{-1} does not always belong to $\mathcal{O}_K$. (this is actually true even for $\mathbb{Q}$)

3.1.5. Classical Hecke characters

Recall that for any fractional ideal $\mathfrak{a}$ one has a decomposition

$$\mathfrak{a} = \prod_{i=1}^{r} \mathfrak{p}_i^{v_{\mathfrak{p}_i}(\mathfrak{a})}$$

Two fractional ideals are said to be coprime, and denote $(\mathfrak{a},\mathfrak{b}) = 1$ if no prime ideal appears in both their decomposition.
Let $\mathfrak{f} \lhd \mathcal{O}_K$ be a non-zero *integral* ideal, then $\mathfrak{f}$ has a unique decomposition into prime ideals $\mathfrak{p}$ in $\mathcal{O}_K$. Consider fractional ideals $\mathfrak{a} = (\alpha)$ generated by units of K. Then one can form the subgroup of $K^\times$ consisting of units that generate fractional ideals co prime to $\mathfrak{f}$, namely

$$K(\mathfrak{f}) = \{\alpha \in K^\times \mid (\mathfrak{a},\mathfrak{f}) = 1\}.$$

Now for $(\beta) := \mathfrak{b}$, the set

$$K(\mathfrak{f})\mathfrak{f} = \{\beta \in K \mid v_{\mathfrak{p}}(\mathfrak{b}) \geq v_{\mathfrak{p}}(\mathfrak{f}) \text{ for all } \mathfrak{p} \text{ appearing in } \mathfrak{f}\} \neq \varnothing$$

as it contains 0 (since $\mathfrak{f}$ does) and for $\beta, \beta' \in K$ that generate $\mathfrak{b}, \mathfrak{b}'$ respectively

$$v_{\mathfrak{p}}(\mathfrak{b}) \geq v_{\mathfrak{p}}(\mathfrak{f}) \quad \text{and} \quad v_{\mathfrak{p}}(\mathfrak{b}') \geq v_{\mathfrak{p}}(\mathfrak{f}) \quad \Rightarrow \quad v_{\mathfrak{p}}(\mathfrak{b}'') \geq v_{\mathfrak{p}}(\mathfrak{f}) \quad \text{where } \mathfrak{b}'' := (b + b') \tag{3.1}$$

Hence one can define the following equivalence relation:

Definition 3.1.6 (Multiplicative congruence)**.** For $\alpha, \beta \in K(\mathfrak{f})$, define the **multiplicative congruence** by the equivalence relation

$$\alpha \equiv \beta \bmod^* \mathfrak{f} \quad \Leftrightarrow \quad \frac{\beta}{\alpha} \in k(\mathfrak{f}) := 1 + k(\mathfrak{f})\mathfrak{f}$$

Remark 3.1.7. (i) Note that $\bmod^*$ is well defined as an equivalence relation since

$$\frac{\beta}{\alpha} \in 1 + k(\mathfrak{f})\mathfrak{f} \quad \Leftrightarrow \quad \beta \in \alpha + \alpha k(\mathfrak{f})\mathfrak{f} \quad \Leftrightarrow \quad \beta - \alpha \in k(\mathfrak{f})\mathfrak{f}$$

which is closed under multiplication by (3.1).

(ii) $K_{\mathfrak{f}} = \{\alpha \in K^\times \mid \alpha \equiv 1 \bmod^* \mathfrak{f}\}$ is a subgroup of $K(\mathfrak{f})$ since

$$\alpha, \beta \equiv 1 \bmod^* \mathfrak{f} \quad \Rightarrow \quad \alpha \equiv \beta \bmod^* \mathfrak{f} \quad \Rightarrow \quad \alpha\beta^{-1} \equiv 1 \bmod^* \mathfrak{f}$$

Moreover, one has an isomorphism

$$\left({}^{\mathcal{O}_K}\!/_{\mathfrak{f}} \right)^\times \cong {}^{K(\mathfrak{f})}\!/_{K_{\mathfrak{f}}} \tag{3.2}$$

The isomorphism is given by the map $\alpha + \mathfrak{f} \longmapsto \alpha + K(\mathfrak{f})\mathfrak{f}$ which is surjective (by the Chinese remainder theorem) and has a kernel equal to $1 + \mathfrak{f}$ (since $\alpha \in \mathcal{O}_K$). We choose the following set of notations for simplicity:

Notation 3.1.8. For a non-zero integral ideal $\mathfrak{f}$ of $\mathcal{O}_K$:

$$\begin{aligned} I(\mathfrak{f}) &:= \{\mathfrak{a} \text{ fractional ideals of } K \mid (\mathfrak{a},\mathfrak{f}) = 1\} \\ P(\mathfrak{f}) &:= \{ \text{ principal fractional ideals } \mathfrak{a} = (\alpha) \in I(\mathfrak{f})\} \\ P_{\mathfrak{f}} &:= \{\mathfrak{a} = (\alpha) \in P(\mathfrak{f}) \mid \alpha \equiv 1 \bmod^* \mathfrak{f}\} \end{aligned}$$

Recall that one has a map

$$\begin{aligned} K &\longrightarrow \mathbb{R}\otimes_{\mathbb{Q}} K \cong (\mathbb{R})^{r_1}\times(\mathbb{C})^{r_2} \\ \alpha &\longmapsto 1\otimes\alpha \end{aligned}$$

where r_1, r_2 is the number of real and respectively complex places of K.

Definition 3.1.9 (Hecke character). Let $\mathfrak{f}$ be a non-zero ideal of $\mathcal{O}_K$, and $\chi_\infty\colon(\mathbb{R}^\times)^{r_1}\times(\mathbb{C}^\times)^{r_2}\longrightarrow\mathbb{C}^\times$ a continuous homomorphism. A **Hecke character of conductor $\mathfrak{f}$ and infinity type** (r_1, r_2) is a continuous homomorphism

$$\chi : I(\mathfrak{f}) \longrightarrow \mathbb{C}^\times$$

that is entirely determined on $P_\mathfrak{f}$ by χ_∞:

$$\chi(\mathfrak{a}) = \chi_\infty(1\otimes\alpha)^{-1} \text{ for all } \mathfrak{a}\in P_\mathfrak{f}$$

Example 3.1.10. (i) For $K = \mathbb{Q}$, Dirichlet characters are Hecke characters (often called Hecke characters of **finite type**) of infinity type $(1, 0)$.

(ii) One can construct non-Dirichlet Hecke character as follow: Let $s\in\mathbb{C}$ and consider the character

$$\chi_s : I(K)\longrightarrow\mathbb{C}^\times$$

where for all $\mathfrak{a} = (\alpha)\in I(K)$

$$\chi_k(\mathfrak{a}) = |\alpha|^s$$

Then, one has the following commutative diagram

$$\begin{tikzcd} \mathbb{Q}^\times \arrow[r, "\alpha\mapsto(\alpha)"] \arrow[d, "\alpha \mapsto 1\otimes\alpha"'] & I(\mathbb{Q}) \arrow[d, "\chi_s"] \\ \mathbb{R}^\times \arrow[r, "\alpha\mapsto|\alpha|^{-s}"'] & \mathbb{C}^\times \end{tikzcd}$$

Thus, χ_s is a well defined Hecke character of conductor $\mathfrak{f} = \mathbb{Z}$ and infinity type $(1, 0)$ given by

$$\begin{aligned} \chi_{s,\infty} : \mathbb{R}^\times &\longrightarrow \mathbb{C}^\times \\ \alpha &\longmapsto |\alpha|^{-s}. \end{aligned}$$

(iii) For $K = \mathbb{Q}(i)$, one can (similarly) construct a family of finite order Hecke characters as follow: Define for any integer k

$$\chi_k : I(K)\longrightarrow\mathbb{C}^\times \tag{3.3}$$

where for all $\mathfrak{a} = (\alpha)\in I(K)$

$$\chi_k(\mathfrak{a}) = \left(\frac{\alpha}{|\alpha|}\right)^{4k}$$

Then, one has the commutative diagram

$$\begin{tikzcd} K^\times \arrow[r, "\alpha\mapsto(\alpha)"] \arrow[d, "\substack{\alpha \\ \downarrow \\ 1\otimes\alpha}"'] & I(K) \arrow[d, "\chi_k"] \\ \mathbb{C}^\times \arrow[r, "\alpha\mapsto\left(\frac{\alpha}{|\alpha|}\right)^{-4k}"'] & \mathbb{C}^\times \end{tikzcd}$$

Thus, χ_k is a well defined Hecke character of conductor $\mathfrak{f} = \mathcal{O}_K$ and infinity type $(0,1)$ defined by

$$\begin{aligned} \chi_{k,\infty}(\alpha) : \mathbb{C}^\times &\longrightarrow \mathbb{C}^\times \\ \alpha &\longmapsto \left(\frac{\alpha}{|\alpha|}\right)^{-4k} \end{aligned}$$

Remark 3.1.11. Let $n = \left| {K(\mathfrak{f})}/{K_\mathfrak{f}} \right| = \left| \left({\mathcal{O}_K}/{\mathfrak{f}} \right)^\times \right| < \infty$.
For any $\alpha \in K^\times$:

$$\alpha \in K(\mathfrak{f}) \quad \Rightarrow \quad \alpha^n \in K_\mathfrak{f} \quad \Rightarrow \quad \chi(\mathfrak{a})^n = \chi_\infty^{-1}(\alpha)^n \quad \Rightarrow \quad \chi(\mathfrak{a}) = \epsilon(\alpha)\chi_\infty^{-1}(\alpha)$$

where $\epsilon(\alpha)$ is a root of unity for all $\alpha \in K^\times$. This defines a (unitary) character

$$\epsilon : \left({\mathcal{O}_K}/{\mathfrak{f}} \right)^\times \longrightarrow \mathbb{S}^1 \tag{3.4}$$

as $\epsilon(\alpha) = \chi(\mathfrak{a})\chi_\infty(\alpha)$ and ϵ is trivial on $K_\mathfrak{f}$.

Hence, given a non-zero integral ideal $\mathfrak{f}$ and a character as in (3.4), one has an equivalent definition of Hecke characters, where now ϵ and χ_∞ determine the Hecke character χ on principal ideals in $P(\mathfrak{f})$ rather than just the ones in $P_\mathfrak{f}$, i.e. one has the following commutative diagram

$$\begin{tikzcd} K(\mathfrak{f}) \arrow[r, "\alpha\longmapsto\mathfrak{a}=(\alpha)"] \arrow[d, "\substack{\alpha \\ \downarrow \\ (\alpha K_\mathfrak{f}, 1\otimes\alpha)}"'] & P(\mathfrak{f}) \arrow[d, "\chi"] \\ {K(\mathfrak{f})}/{K_\mathfrak{f}} \times (\mathbb{R}^\times)^{\sigma_1} \times (\mathbb{C}^\times)^{\sigma_2} \arrow[r, "\epsilon\cdot\chi_\infty"'] & \mathbb{C}^\times \end{tikzcd}$$

In (ii) of example 3.1.10, the ϵ-character is trivial.

3.1.12. Classification of local characters

Let K_v be the completion of K with respect to a place v and $|.|_v$ the normalised absolute value on the completion K_v. A **local character** χ_v relative to a place v is a continuous homomorphism

$$\chi_v : K_v^\times \longrightarrow \mathbb{C}^\times$$

It is said to be **unramified** if it is trivial on the local units, i.e.

$$\chi_{|\mathcal{O}_v^\times} = 1$$

(For example, if $K = \mathbb{R}$, this translates into $\chi(-1) = 1$)

Since $K_v^\times \cong \mathcal{O}_v^\times \times |K_v^\times|_v$, every character factorises as a product $\chi = \chi' \cdot \chi''$ of characters on $\mathcal{O}_v^\times$ and $|K_v^\times|_v$ respectively, where

(i) Compactness of $\mathcal{O}_v^\times$ makes all characters χ' unitary (these characters are actually pullbacks of unitary characters, defined by the restriction of χ).

(ii) Characters χ'' on $|K_v^\times| = \{y \in \mathbb{R}_{>0} \,/\, |\alpha|_v \text{ for some } \alpha \in K_v^\times\}$ are all of the form

$$\chi'' : y \longmapsto y^s = |\alpha|_v^s \quad \text{for some } \alpha \in K_v^\times, s \in \mathbb{C}$$

In particular,

$$\chi_v \text{ is unramified} \quad \Leftrightarrow \quad \chi = |\,.\,|_v^s \quad \text{for some } s \in \mathbb{C}$$

This allows us to classify all the local characters in the archimedean case:

(i) $K_v = \mathbb{R}$: $\mathcal{O}_v^\times = \{\pm 1\}$ and the only unitary characters of $\mathbb{R}^\times$ are

$$\chi' : \alpha \longmapsto 1 \quad \text{and} \quad \chi' : \alpha \longmapsto \operatorname{sgn}(\alpha) = \frac{\alpha}{|\alpha|}$$

Thus

$$\chi_v(x) = \begin{cases} |x|^s & \text{if } \chi_v \text{ is unramified} \\ \operatorname{sgn}(x)|x|^s & \text{if } \chi_v \text{ is ramified} \end{cases}$$

(ii) $K_v = \mathbb{C}$: $\mathcal{O}_v^\times = \mathbb{S}^1$ and the only unitary characters of $\mathbb{C}^\times$ are

$$\chi_n' : e^{i\theta} \longmapsto e^{in\theta} \quad \text{for some } n \in \mathbb{Z}$$

Thus

$$\chi_{n,s}(z) = \chi_{n,s}(re^{i\theta}) = r^s e^{in\theta} = |z|^s \operatorname{Arg}(z^n)$$

We end this subsection with the following useful remark:

Remark 3.1.13. The kernel of any local character is an open subgroup of $K_v^\times$:
Indeed, by continuity, $\chi(\mathcal{O}_v^\times) \leq \mathbb{S}^1$ since it is compact (otherwise, it will not be bounded). Let $U_x = \{e^{2\pi i x} \mid -\pi < x < \pi\} \subset \mathbb{S}^1$, then $\chi^{-1}(U_x) \cap \mathcal{O}_v^\times$ is an open neighbourhood of 1, hence it contains a subgroup $1 + \pi^n \mathcal{O}_v^\times$ for some $n \geq 1$. But then, $\chi(1 + \pi^n \mathcal{O}_v^\times)$ is a subgroup of $\mathbb{S}^1$ contained in U and the only such possible subgroup is $\{1\}$. Hence it must lay in the kernel.

3.1.14. Idèlic Hecke characters

Recall that for each finite set S of places (i.e. valuations on K) that includes all the infinite ones (valuations coming from real/complex embeddings), one can form the topological product

$$\mathbb{A}_S = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v \quad \text{and} \quad \mathbb{J}_S = \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$$

The Adèlic (Idèlic) topology is defined to be the colimit topology given by

$$\mathbb{A} = \varprojlim_S \mathbb{A}_S \quad \text{and} \quad \mathbb{J}_K = \varprojlim_S \mathbb{J}_S$$

Notation 3.1.15. We will write $v \mid \infty$ for infinite places and $v \nmid \infty$ for finite places, and adopt the following notations:

$$\begin{aligned} &\mathbb{J}_\infty = (K \otimes_{\mathbb{Q}} \mathbb{R})^\times = (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}, \quad \mathbb{J}_\infty^+ = (\mathbb{R}_{>0})^{r_1} \times (\mathbb{C}^\times)^{r_2} \\ &\text{and} \quad \mathbb{J}_K = \textstyle\prod_v K_v^\times \qquad \text{for almost all } v. \end{aligned} \tag{3.5}$$

Definition 3.1.16 (Idèlic Hecke character)**.** A **Hecke character** of a number field K is a continuous (with respect to the idèlic topology) homomorphism

$$\chi : \mathbb{J}_K \longrightarrow \mathbb{C}^\times$$

Such that $\chi(K^\times) = 1$.

Remark 3.1.17. (i) Note that χ is trivial on (the image of) $K^\times$ through the embedding

$$\begin{aligned} K^\times &\hookrightarrow \mathbb{J}_K \\ \alpha &\longmapsto (\alpha, \alpha, \alpha, \dots) \end{aligned}$$

which makes sense since $|\alpha|_v = 1$ for almost all places v of K.

(ii) The isomorphism in (3.2) in the case of K_v becomes

$$\left(\mathcal{O}_v / \mathfrak{p}^{e_v}\right)^\times \cong \mathcal{O}_v^\times / 1 + \mathfrak{p}^{e_v}$$

(iii) The **idèle class group** of K is the topological group

$$\mathfrak{Cl}(K) = \mathbb{J}_K / K^\times$$

A Hecke character is sometimes also called an **Idèle class character** and can be equivalently defined as a continuous homomorphism

$$\chi : \mathfrak{Cl}(K) \longrightarrow \mathbb{C}^\times$$

Proposition 3.1.18. *One has a bijection*

$$\begin{aligned}\{\textit{Characters of } \mathbb{J}_K\} &\xrightarrow{\sim} \{(\chi_v)_v \,|\, \chi_v : K_v^\times \to \mathbb{C}^\times \textit{ is unramified for almost all } v\}\\ \chi &\longmapsto (\chi_v)_v\end{aligned}$$

where, for each place v of K, the local character χ_v is defined as

$$\begin{aligned}\chi_v = \chi_{|K_v^\times} : K_v^\times &\longrightarrow \mathbb{C}^\times\\ \alpha_v &\longmapsto \chi(1,\dots,\underset{\substack{\uparrow\\ v\textit{th component}}}{\alpha_v},\dots,1)\end{aligned}$$

Proof. Note that this map is well defined since the profinite completion

$$\widehat{\mathcal{O}}_v = \prod_{v\nmid\infty} \mathcal{O}_v^\times$$

is profinite. By the same argument as in remark 3.1.13, $\ker\left(\chi_{|\widehat{\mathcal{O}}_K}\right)$ contains an open subgroup of $\prod_{v\nmid\infty} \mathcal{O}_v^\times$, namely

$$W = \prod_{v\in S}(1+\pi_v^n\mathcal{O}_v) \times \prod_{v\notin S}\mathcal{O}_v^\times \qquad \text{for some finite } S, n \geq 1$$

If one considers the composition

$$\chi_v \hookrightarrow \mathbb{J}_K \xrightarrow{\chi} \mathbb{C}^\times$$

then χ_v is continuous for all places v and for finitely many $v \notin S$:

$$\chi_v(\mathcal{O}_v) = 1$$

On the other hand, given a family $(\chi_v : K_v^\times \longrightarrow \mathbb{C}^\times)_v$ one defines the character

$$\begin{aligned}\chi = \prod_v \chi_v : \mathbb{J}_K &\longrightarrow \mathbb{C}^\times\\ (\alpha_v)_v &\longmapsto \prod_v \chi_v(\alpha_v)\end{aligned}$$

and by assumptions $\chi_v(\alpha_v) = 1$ for almost all v. $\square$

Remark 3.1.19. As seen above, an idèlic Hecke character $\chi = \prod_v \chi_v$ comes always with a conductor

$$\mathfrak{f} = \prod_v \mathfrak{p}_v^{e_v}$$

(i) One can recover a classical Hecke character from an idèlic one as follow: Define a character

$$\widetilde{\chi} : I(\mathfrak{f}) \longrightarrow \mathbb{C}^\times$$

with $\widetilde{\chi}(\mathfrak{p}_v) = \chi_v(\pi_v)$ for all non-archimedean $v \nmid \mathfrak{f}$.

In order to recover a classical Hecke character, one needs an infinity type $\widetilde{\chi}_\infty$ on $\mathbb{J}_\infty$ such that the following commutes:

$$\begin{tikzcd} K_{\mathfrak{f}} \arrow[r] \arrow[d] & I(\mathfrak{f}) \arrow[d, "\widetilde{\chi}"] \\ \mathbb{J}_\infty \arrow[r, "\widetilde{\chi}_\infty"'] & \mathbb{C}^\times \end{tikzcd}$$

For $\alpha \in K_{\mathfrak{f}}$ one has the composition

$$\alpha \longmapsto (\alpha) = \prod \mathfrak{p}_v^{e'_v} \in I(\mathfrak{f}) \longmapsto \prod_v \widetilde{\chi}(\mathfrak{p}_v)^{e'_v}.$$

On the other hand, since χ vanishes on $K^\times$,

$$1 = \chi(\alpha) = \prod_{v|\infty} \chi_v(\alpha) \prod_{v\nmid\infty} \chi_v(\alpha) = \chi_\infty(\alpha) \prod_v \chi_v(\mathcal{O}_v^\times \pi_v)^{e'_v} = \chi_\infty(\alpha) \prod_v \widetilde{\chi}(\mathfrak{p}_v)^{e'_v}$$

$$\Rightarrow \quad \chi_\infty^{-1}(\alpha) = \prod_v \widetilde{\chi}(\mathfrak{p}_v)^{e'_v}$$

Using the identification in (3.5), one sees that

$$\alpha \longmapsto \alpha \otimes 1 \in \mathbb{J}_\infty \longmapsto \chi_\infty^{-1}(\alpha \otimes 1) = \prod_v \widetilde{\chi}(\mathfrak{p}_v)^{e'_v}$$

Hence for any given idèlic character χ, the corresponding character $\widetilde{\chi}$ on $P_{\mathfrak{f}}$ is a classical Hecke character, with conductor $\mathfrak{f}$ and infinity type $\widetilde{\chi}_\infty = \chi_\infty^{-1}(\alpha)$.

(ii) The converse is also true: A classical Hecke character can also be seen as an idèlic character. We illustrate that for the simple case of a Dirichlet character: Let $K = \mathbb{Q}$, and $\mathbb{J}_{\mathbb{Q}}$ be the group of idèles on $\mathbb{Q}$. The p-adic valuation induces a canonical isomorphism

$$\mathbb{Q}_p^\times \cong \langle p \rangle \times \mathbb{Z}_p^\times$$

And one has

$$\begin{aligned} \mathbb{J}_{\mathbb{Q}} &= \mathbb{R}^\times \times \prod_p \mathbb{Q}_p^\times \qquad \text{for almost all } p \\ &\cong (\{\pm 1\} \times \mathbb{R}_{>0}) \times \left(\prod_p \mathbb{Z}_p^\times \times \bigoplus_p \mathbb{Z} \right) \\ &\cong \left(\{\pm 1\} \times \bigoplus_p \mathbb{Z} \right) \times \mathbb{R}_{>0} \times \prod_p \mathbb{Z}_p^\times \\ &\cong \mathbb{Q}^\times \times \mathbb{R}_{>0} \times \prod_p \mathbb{Z}_p^\times \end{aligned}$$

Now, for a Dirichlet character

$$\chi : G_n \longrightarrow \mathbb{S}^1,$$

The Chinese Remainder Theorem for $n = \prod_p p^{e_p}$ implies that

$$\mathbb{Z}/_{n\mathbb{Z}} \quad \cong \quad \bigoplus_p \left(\mathbb{Z}/_{p^{e_p}\mathbb{Z}}\right)$$

Hence, taking the inverse limit in both sides

$$\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/_{n\mathbb{Z}} \quad \cong \quad \prod_p \varprojlim_m \left(\mathbb{Z}/_{p^m\mathbb{Z}}\right) \quad \cong \quad \prod_p \mathbb{Z}_p$$

and one gets a decomposition

$$\chi = \prod_p \chi_p : \prod_p \mathbb{Z}_p^\times \cong \widehat{\mathbb{Z}}^\times \longrightarrow \mathbb{C}^\times$$

Which, idèlicaly, can be viewed as a continuous character

$$\chi_{Hecke} : \mathbb{J}_\mathbb{Q} \longrightarrow \mathbb{C}^\times$$

with $\chi_{Hecke}(\alpha, u, r) = \chi(u)$.

3.2. Hecke L-functions

The idea, as in theorem 3.1.4, would be to construct a global L-function as an Euler product of local L-functions, i.e. entities involving local characters $\chi_v : K_v^\times \longrightarrow \mathbb{C}^\times$.

3.2.1. Algebraic Hecke characters

From now on, we will consider the idèlic definition of Hecke characters.

Definition 3.2.2 (Algebraic homomorphism)**.** Let K be a number field. A homomorphism $\varphi : K^\times \longrightarrow \mathbb{C}^\times$ is said to be **algebraic** if for every embedding $\sigma : K \hookrightarrow \mathbb{C}$ there exists an integer n_σ such that for all $x \in K$

$$\varphi(x) = \prod_\sigma \sigma(x)^{n_\sigma}$$

Note that each real (resp. complex) embedding $\sigma_v : K \hookrightarrow K_v \cong \mathbb{R}$ corresponds to a real (resp. complex) place v. One can then extend an algebraic homomorphism φ to

$$\begin{aligned} \varphi : \mathbb{J}_\infty &\longrightarrow \mathbb{C}^\times \\ x_v &\longmapsto \prod_{v \text{ real}} x_v^{n_{\sigma_v}} \cdot \prod_{v \text{ complex}} x_v^{n_{\sigma_v}} x_v^{n_{\overline{\sigma}_v}} \end{aligned}$$

Definition 3.2.3 (Algebraic Hecke character)**.** A Hecke character $\chi : \mathbb{J}_K \longrightarrow \mathbb{C}^\times$ is said to be **algebraic** if there exists an algebraic homomorphsim $\varphi : K^\times \longrightarrow \mathbb{C}^\times$ such that

$$\varphi(\alpha) = \chi_\infty(\alpha) \quad \text{for all } \alpha \in \mathbb{J}_\infty^+$$

Where $\chi_\infty = \prod_{v|\infty} \chi_v : J_\infty \longrightarrow \mathbb{C}^\times$.
More precisely, this means that there exists integers a_v, b_v and b'_v such that

(i) if v is a real place, $\chi_v(x) = x^{a_v}$ for all $x \in \mathbb{R}_{>0}$.

(ii) if v is a complex place, $\chi_v(z) = z^{b_v}\overline{z}^{b'_v}$ for all $z \in \mathbb{C}^\times$.

Example 3.2.4. (i) Consider the Norm map

$$\begin{aligned} N_{K/\mathbb{Q}} : K^\times &\longrightarrow \mathbb{C}^\times \\ \alpha &\longmapsto \prod_\sigma \sigma(\alpha) \end{aligned}$$

It is an algebraic homomorphism and for $K = \mathbb{Q}$, then $N_{\mathbb{Q}/\mathbb{Q}} = id$ and $n_{\sigma_v} = 1$. Recall that the idèlic absolute value

$$\begin{aligned} |\,.\,|_{\mathbb{A}} : \mathbb{J}_K &\longrightarrow \mathbb{R}_{>0} \\ (x_v)_v &\longmapsto \prod_v |x_v|_v \end{aligned}$$

defines a Hecke character as the product formula implies that $|K^\times|_{\mathbb{A}} = \{1\}$. Moreover, one has

$$\chi_\infty = \prod_{v|\infty} |\cdot|_v$$

(ii) For $K = \mathbb{Q}(i)$ the family of Hecke character defined in (3.3) are algebraic of type $(0, 1)$.

(iii) In general, one can show that for every totally real field K (i.e. a field that has no complex place), algebraic Hecke characters are all of the form

$$\chi(\alpha) = \chi_{fin}(\alpha) \cdot |\alpha|_{\mathbb{A}}^n$$

where the integer n does not depend on v, and χ_{fin} is a Dirichlet character.

3.2.5. Local L-function associated to local Hecke characters

Recall from §3.1.12 that for a number field K one can classify all local characters χ_v where v is archimedean. We start thus by defining the **local L-functions** corresponding to such local characters:

(i) For $K_v = \mathbb{R}$:

$$L(s,\chi_v) := L(\chi'|\,.\,|^s) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) & \text{if } \chi_v \text{ is unramified} \\ \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right) & \text{if } \chi_v \text{ is ramified} \end{cases}$$

(ii) For $K_v = \mathbb{C}$:

$$L(s,\chi_v) := L(\chi_{s,n}) = (2\pi)^{-\left(s+\frac{|n|}{2}\right)}\Gamma\left(s+\frac{|n|}{2}\right)$$

When K_v is a p-adic field, we define the local L-function corresponding to χ_v to be

$$L(s,\chi_v) := \begin{cases} \frac{1}{1-\chi_v(\pi_v)} & \text{if } \chi_v \text{ is unramified} \\ 1 & \text{if } \chi_v \text{ is ramified} \end{cases}$$

Proposition 3.2.6 (Global L-functions)**.** *Let $\chi = \prod_v \chi_v$ be a Hecke character. We define the L-function associated to the Hecke character χ to be*

$$L(s,\chi) := \prod_v L(s,\chi_v) \qquad \textit{which converges absolutely for } \Re(s) > 1$$

Proof. Let $\chi = \prod_v \chi_v = \chi' \cdot |\,.\,|_{\mathbb{A}}^s$ for some $s \in \mathbb{C}$. By proposition 3.1.18, χ_v is unramified for almost all v hence one can ignore the finite set of places where it is ramified (and where $L(s,\chi_v) = 1$). Observe that

$$\begin{aligned}\prod_v |L(s,\chi_v)| = \prod_v \frac{1}{|1-\chi_v(\pi_v)|} &= \prod_v \frac{1}{|1-\chi'_v(\pi_v)|\pi_v|^s|} \\ &= \prod_v \frac{1}{|1-\chi'_v(\pi_v)q_v^{-s}|}\end{aligned}$$

where $q_v = \left|{\mathcal{O}_v}/{(\pi_v)}\right|$. Hence, taking the log, one has

$$\begin{aligned}\log\left(\prod_v |L(s,\chi_v)|\right) = \sum_v \log\left(\frac{1}{|1-\chi'_v(\pi_v)q_v^{-s}|}\right) &= \sum_v \Re\left(\log\frac{1}{1-\chi'_v(\pi_v)q_v^{-s}}\right) \\ &= \Re\left(\sum_v \log\frac{1}{1-\chi'_v(\pi_v)q_v^{-s}}\right)\end{aligned}$$

Using the power series expansion of $\log\left(\frac{1}{1-x}\right)$ one gets

$$\log\left(\prod_v |L(s,\chi_v)|\right) = \Re\left(\sum_v\sum_k \frac{\chi'_v(\pi_v)^k q_v^{-ks}}{k}\right)$$

As χ'_v is unitary and $\Re(q_v^{-ks}) \leq q_v^{-k\Re(s)}$, it suffices to show that the sum

$$\sum_v \sum_{k\geq 1} \frac{q_v^{-k\Re(s)}}{k} \quad \text{converges for all } \Re(s) > 1$$

Now, recall that for all $v \nmid \infty$ in K, one has that $v \mid p$ for some prime p and $q_v = \left| {}^{O_v}\!/_{\pi_v} \right| = \left| {}^{\mathbb{Z}_p}\!/_{p\mathbb{Z}_p} \right|^{e_v} = p^{e_v}$ with $e_v \geq 1$. Thus, one gets that

$$\begin{aligned}
\sum_v \sum_{k\geq 1} \frac{q_v^{-k\Re(s)}}{k} &= \sum_{v|p} \sum_p \sum_{k\geq 1} \frac{q_v^{-k\Re(s)}}{k} = \sum_{v|p} \sum_p \sum_{k\geq 1} \frac{p^{-ke_v\Re(s)}}{k} \\
&\leq \sum_{v|p} \sum_p \sum_{k\geq 1} \frac{p^{-k\Re(s)}}{k} \\
&\leq n \sum_p \sum_{k\geq 1} \frac{p^{-k\Re(s)}}{k} \quad \text{since the number of places } v \mid p \text{ is } \leq [K:\mathbb{Q}] := n \\
&= n \log\left(\prod_p \frac{1}{1-p^{-\Re(s)}} \right) = n \log\left(\sum_{k\geq 1} k^{-\Re(s)} \right)
\end{aligned}$$

Where the Euler product attached to the Riemann ζ-function

$$\prod_p \frac{1}{1-p^{-\Re(s)}} = \prod_p \left(\sum_{k\geq 0} p^{-k\Re(s)} \right) = \sum_{k\geq 1} \frac{1}{k^{\Re(s)}}$$

converges absolutely for $\Re(s) > 1$. □

Definition 3.2.7 (Hecke L-functions)**.** Let χ be a Hecke character. For a complex number $s \in \mathbb{C}$, the **global Hecke L-function** associated to χ is defined to be

$$L(s,\chi) = \prod_{v\nmid\infty} L(s,\chi_v) = \prod_{v\nmid\infty} \frac{1}{1-\chi_v(\pi_v)q_v^{-s}}$$

One also defines

$$\Lambda(s,\chi) = \prod_{v|\infty} L(s,\chi_v)$$

where

$$L(\chi\cdot|.|^s) = L(s,\chi)\Lambda(s,\chi)$$

Example 3.2.8. The Riemann and Dedekind ζ-functions are particular examples of Hecke L-functions: Let χ be the trivial Hecke character, thus all the local characters are trivial for all places v of K.

(i) For $K = \mathbb{Q}$:

$$\begin{aligned}
L(s,\chi) = \prod_{v\nmid\infty} L(s,\chi_v) &= \prod_{v\nmid\infty} \frac{1}{|1-|\pi_v|_v^s|} = \prod_{v\nmid\infty} \frac{1}{1-|\pi_v|_v^s} \\
&= \prod_p \frac{1}{1-q_p^{-s}} = \prod_p \frac{1}{1-p^{-s}} = \zeta(s)
\end{aligned}$$

(ii) Analogously, for an arbitrary number field K:

$$L(s,\chi)=\sum_{0\neq\mathfrak{a}\triangleleft\mathcal{O}_K}\frac{1}{N(\mathfrak{a})^s}=\zeta_K(s)$$

Remark 3.2.9. Recall that from remark 3.1.19, one can always recover a classical Hecke character from an idèlic one by setting

$$\widetilde{\chi}(\mathfrak{p}_v)=\chi_v(\pi_v)\qquad\text{for all non-archimedean } v\nmid\mathfrak{f}$$

Hence, the Hecke L-function is given by the L-series

$$\begin{aligned}L(s,\chi)&=\prod_{v\nmid\infty}L(s,\chi_v)=\prod_{v\nmid\mathfrak{f}}\frac{1}{1-\chi_v(\pi_v)q_v^{-s}}=\prod_{v\nmid\mathfrak{f}}\frac{1}{1-\widetilde{\chi}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-s}}\\&=\sum_{\substack{0\neq\mathfrak{a}\triangleleft\mathcal{O}_K\\(\mathfrak{a},\mathfrak{f})=1}}\frac{\widetilde{\chi}(\mathfrak{a})}{N(\mathfrak{a})^s}\end{aligned}$$

In his thesis, Tate proved that for a unitary Hecke character χ, the function $L(\chi\cdot|\,.\,|^s)$ (which is holomorphic on $\{s\in\mathbb{C}\,|\,\mathfrak{R}(s)>1\}$) admits a meromorphic continuation to the whole s-plane with only possible poles at $s=i\lambda$ and $s=1+i\lambda$ when $\chi=|\,.\,|^{-i\lambda}$, $\lambda\in\mathbb{R}$.

Theorem 3.2.10 (Hecke). *The Hecke L-function associated to a Hecke character $\chi=(\chi_v)_v$ with conductor $\mathfrak{f}$ on a number field K, admits an analytic continuation to the complex plane and satisfies the functional equation*

$$\Lambda(s,\chi)=\epsilon(s,\chi)\Lambda(1-s,\chi^\vee)\tag{3.6}$$

Where $\chi^\vee=\chi^{-1}|\,.\,|$, Δ_K is the discriminant of K, $\epsilon(s,\chi)$ a global factor given by $\epsilon(s,\chi)=\prod_v\epsilon_v(\chi_v|\,.\,|^s)\in\mathbb{C}^\times$ and

$$\begin{aligned}\Lambda(s,\chi)=L(s,\chi)\cdot\left(N_{\mathbb{F}/\mathbb{Q}}(\mathfrak{f})|\Delta_K|\right)^{\frac{s}{2}}&\left(\prod_{\textit{complex } v}(2\pi)^{-\left(s+\frac{|n|}{2}\right)}\Gamma\left(s+\frac{|n|}{2}\right)\right)\\&\left(\prod_{\substack{\textit{real } v\\ \chi_v\ \textit{ramified}}}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)\left(\prod_{\substack{\textit{real } v\\ \chi_v\ \textit{unramified}}}\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right)\end{aligned}$$

Proof. For a proof, see [Tat67]. □

3.2.11. Special values of Hecke L-functions on imaginary quadratic fields

Let K be an imaginary quadratic number field, $\mathcal{O}_K$ its ring of integers and i_∞ a fixed complex embedding so all integral ideals of $\mathcal{O}_K$ are realised as lattices in $\mathbb{C}$. Recall that by remark 3.1.11, an algebraic (classical) Hecke character χ on K with conductor $\mathfrak{f}$ and infinity type (a,b) is of the form

$$\chi(\mathfrak{a}) = \epsilon(\alpha)\chi_\infty^{-1}(\alpha) = \epsilon(\alpha)\cdot\alpha^a\overline{\alpha}^b$$

where $\mathfrak{a} = (\alpha)$ for all $\alpha \in K(\mathfrak{f})$, and has an associated *Hecke* L-series

$$L_\mathfrak{f}(s,\chi) = \sum_{\substack{0\neq\mathfrak{a}\triangleleft\mathcal{O}_K\\(\mathfrak{a},\mathfrak{f})=1}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}$$

which is absolutely convergent on $\{z\in\mathbb{C}\,|\,\Re(s) > \frac{a+b}{2}+1\}$.
Consider the **Ray class group**

$$C_\mathfrak{f}(K) := {}^{I(\mathfrak{f})}\big/_{P_\mathfrak{f}}$$

and let $\{\mathfrak{a}_1,\dots,\mathfrak{a}_h\}$ be a set of representatives, where h is the class number of K (which is finite). For an integer $a\in\mathbb{N}$ and $s>\frac{a}{2}+1$

$$L_\mathfrak{f}(s,\overline{\chi}^a) = \sum_{\mathfrak{a}\in C_\mathfrak{f}(K)} \frac{\overline{\chi}(\mathfrak{a})^a}{N(\mathfrak{a})^s} = \sum_{i=1}^h \sum_{\substack{0\neq\mathfrak{b}\triangleleft\mathcal{O}_K\\ \mathfrak{b}\sim\mathfrak{a}_i}} \frac{\overline{\chi(\mathfrak{b})}^a}{N(\mathfrak{b})^s}$$

Observe that, since

$$\mathfrak{b}\sim\mathfrak{a}_i \Leftrightarrow \mathfrak{b} = \beta\mathfrak{a}_\mathrm{i} \quad \text{where} \quad 0\neq\beta\in{\mathfrak{a}_\mathrm{i}}^{-1} = \{\beta\in K\,|\,\beta\mathfrak{a}_i\triangleleft\mathcal{O}_K\},$$

one has a bijection

$$\begin{aligned}\{0\neq\mathfrak{b}\triangleleft\mathcal{O}_K\,:\,\mathfrak{b}\sim\mathfrak{a}_i\} &\xrightarrow{\sim} {}^{\{0\neq\beta\in\mathfrak{a}_i^{-1}\,:\,\beta\equiv 1 \bmod^*\mathfrak{f}\}}\big/_{\mathcal{O}_\mathfrak{f}^\times} := \{u\in\mathcal{O}_K^\times\mid u\equiv 1 \bmod^*\mathfrak{f}\}\\ \mathfrak{b} &\longmapsto \beta\mathfrak{b}\end{aligned}$$

Thus

$$\begin{aligned}L_\mathfrak{f}(s,\overline{\chi}^a) &= \frac{1}{\omega_\mathfrak{f}}\sum_{i=1}^h \sum_{\substack{0\neq\beta\in\mathfrak{a}_i^{-1}\\ \beta\equiv 1 \bmod^*\mathfrak{f}}} \frac{\overline{\chi(\beta\mathfrak{a}_i)}^a}{N(\beta\mathfrak{a}_\mathrm{i})^s} = \frac{1}{\omega_\mathfrak{f}}\sum_{i=1}^h \frac{\overline{\chi(\mathfrak{a}_i)}^a}{N(\mathfrak{a}_i)^s} \sum_{\substack{0\neq\beta\in\mathfrak{a}_i^{-1}\\ \beta\equiv 1 \bmod^*\mathfrak{f}}} \frac{\overline{\chi(\beta)}^a}{|\beta|^{2s}}\\ &= \frac{1}{\omega_\mathfrak{f}}\sum_{i=1}^h \frac{\overline{\chi(\mathfrak{a}_i)}^a}{N(\mathfrak{a}_i)^s} \sum_{\substack{0\neq\beta\in\mathfrak{a}_i^{-1}\\ \beta\equiv 1 \bmod^*\mathfrak{f}}} \frac{\overline{\beta}^a}{|\beta|^{2s}}\end{aligned}$$

where $\omega_\mathfrak{f} = \left|\mathcal{O}_\mathfrak{f}^\times\right|$. Note that one has a bijection

$$\begin{aligned}\mathfrak{f}\mathfrak{a}_i^{-1} &\xrightarrow{\sim} \{\beta\in\mathfrak{a}_i^{-1}\,:\,\beta\equiv 1 \bmod^*\mathfrak{f}\}\\ \gamma &\longmapsto \gamma+\alpha_i \qquad \text{where } \alpha_i\in\mathfrak{a}_i^{-1}\,;\,\alpha_i\equiv 1 \bmod^*\mathfrak{f}\end{aligned}$$

Hence

$$
\begin{aligned}
L_{\mathfrak{f}}(s,\overline{\chi}^a) &= \frac{1}{\omega_{\mathfrak{f}}}\sum_{i=1}^{h}\frac{\overline{\chi(\mathfrak{a}_i)}^a}{N(\mathfrak{a}_i)^s}\sum_{\gamma\in\mathfrak{f}\mathfrak{a}_i^{-1}}\frac{(\overline{\gamma+\alpha_i})^a}{|\gamma+\alpha_i|^{2s}} \\
&= \frac{1}{\omega_{\mathfrak{f}}}\sum_{i=1}^{h}\sum_{\gamma\in\mathfrak{f}\mathfrak{a}_i^{-1}}\frac{\overline{\chi(\mathfrak{a}_i)}^a}{N(\mathfrak{a}_i)^s}\frac{(\overline{\gamma+\chi(\alpha_i)})^a}{|\gamma+\chi(\alpha_i)|^{2s}} \qquad \text{since } \chi \text{ is algebraic of } \infty\text{-type } (1,0) \\
&= \sum_{i=1}^{h}\sum_{\gamma\in\mathfrak{f}\mathfrak{a}_i^{-1}}\frac{\left(\overline{\chi(\alpha_{\mathfrak{a}}\mathfrak{a})}+\overline{\chi(\mathfrak{a})\gamma}\right)^a}{|\chi(\alpha_{\mathfrak{a}}\mathfrak{a})+\chi(\mathfrak{a})\gamma|^{2s}}
\end{aligned}
$$

Here we used the fact that $N(\mathfrak{a}_i)=|\chi(\mathfrak{a}_i)|^2$ and $\omega_{\mathfrak{f}}=1$. Indeed, for the first assertion, observe that the following Hecke character is of finite order (has ∞-type $(0,0)$)

$$
\varphi:\mathfrak{a}\in C_{\mathfrak{f}}(K)\longmapsto \frac{|\chi(\mathfrak{a})|^2}{N(\mathfrak{a})} = \underset{\underset{(1,0)}{\uparrow}}{\chi(\mathfrak{a})}\,\overset{\overset{(0,1)}{\downarrow}}{\overline{\chi}(\mathfrak{a})}\,\underset{\underset{(-1,-1)}{\uparrow}}{N(\mathfrak{a})^{-1}}\in\mathbb{R}_{>0}
$$

Since the only finite (multiplicative) subgroup of $\mathbb{R}_{>0}$ is $\{1\}$, φ is trivial. $\omega_{\mathfrak{f}}=1$ follows from the fact that χ has ∞-type $(1,0)$.

Thus, this relates the Hecke L-series associated to an algebraic Hecke character to the Eisenstein-Kronecker-Lerch series in (2.3.1) and one gets the following:

Theorem 3.2.12. *Let χ be a Hecke character of conductor $\mathfrak{f}$ and ∞-type $(1,0)$. Then one has*

$$
L_{\mathfrak{f}}(s,\overline{\chi}^a) = \sum_{\mathfrak{a}\in C_{\mathfrak{f}}}K_a^*\left(\chi(\alpha_{\mathfrak{a}}\mathfrak{a}),0,s;\Lambda\right) = \sum_{\mathfrak{a}\in C_{\mathfrak{f}}}\sum_{\gamma\in\mathfrak{f}\mathfrak{a}^{-1}}\frac{\left(\overline{\chi(\alpha_{\mathfrak{a}}\mathfrak{a})}+\overline{\chi(\mathfrak{a})\gamma}\right)^a}{|\chi(\alpha_{\mathfrak{a}}\mathfrak{a})+\chi(\mathfrak{a})\gamma|^{2s}}
$$

where $\Lambda := \chi(\mathfrak{a})\mathfrak{f}\mathfrak{a}^{-1}$ and $\alpha_{\mathfrak{a}}\in\mathfrak{a}^{-1}$ such that $\alpha_{\mathfrak{a}}\equiv 1 \bmod^*\mathfrak{f}$.

More generally, we have the following corollary

Corollary 3.2.13. *Let χ be a Hecke character of conductor $\mathfrak{f}$ and ∞-type (a,b) with $b-a>0$. Then one has*

$$
L_{\mathfrak{f}}(s,\chi) = \frac{1}{\omega_{\mathfrak{f}}}\sum_{\mathfrak{a}\in C_{\mathfrak{f}}}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}K_{b-a}^*\left(\alpha_{\mathfrak{a}},0,s-a;\mathfrak{f}\mathfrak{a}^{-1}\right)
$$

where $\omega_{\mathfrak{f}}=\left|\mathcal{O}_{\mathfrak{f}}^{\times}\right|$ and $\alpha_{\mathfrak{a}}\in\mathfrak{a}^{-1}$ such that $\alpha_{\mathfrak{a}}\equiv 1 \bmod^*\mathfrak{f}$. In particular,

$$
L_{\mathfrak{f}}(0,\chi) = \frac{1}{\omega_{\mathfrak{f}}}\sum_{\mathfrak{a}\in C_{\mathfrak{f}}}\chi(\mathfrak{a})e_{a,b}^*\left(\alpha_{\mathfrak{a}},0;\mathfrak{f}\mathfrak{a}^{-1}\right) \tag{3.7}
$$

Proof. By the same reasoning as in the proof of theorem 3.2.12, one gets

$$L_{\mathfrak{f}}(s,\chi) = \frac{1}{\omega_{\mathfrak{f}}}\sum_{\mathfrak{a}\in C_{\mathfrak{f}}}\sum_{\substack{0\neq\beta\in\mathfrak{a}^{-1}\\ \beta\equiv 1 \bmod^{*}\mathfrak{f}}}\frac{\chi(\beta\mathfrak{a})}{N(\beta\mathfrak{a})^{s}} = \sum_{\mathfrak{a}\in C_{\mathfrak{f}}}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^{s}}\sum_{\substack{0\neq\beta\in\mathfrak{a}^{-1}\\ \beta\equiv 1 \bmod^{*}\mathfrak{f}}}\frac{\chi(\beta)}{|\beta|^{2s}}$$

$$= \sum_{\mathfrak{a}\in C_{\mathfrak{f}}}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^{s}}\sum_{\substack{0\neq\beta\in\mathfrak{a}^{-1}\\ \beta\equiv 1 \bmod^{*}\mathfrak{f}}}\frac{\beta^{a}\overline{\beta}^{b}}{|\beta|^{2s}} \quad \text{since } \chi \text{ is algebraic of } \infty\text{-type } (a,b)$$

$$= \sum_{\mathfrak{a}\in C_{\mathfrak{f}}}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^{s}}\sum_{\substack{0\neq\beta\in\mathfrak{a}^{-1}\\ \beta\equiv 1 \bmod^{*}\mathfrak{f}}}\frac{\overline{\beta}^{b-a}}{|\beta|^{2(s-a)}}$$

The result follows in the same manner. □

3.3. Application: the case of a CM elliptic curve

> "*The theory of complex multiplication is not only the most beautiful part of mathematics but also of all science.*"
>
> David Hilbert

3.3.1. Complex multiplication

We go back to our setting in §2.3.1. Let $E(\mathbb{C}) \cong {}^{\mathbb{C}}/_{\Gamma}$ be an elliptic curve with origin $O = \{0\}$, where $\Gamma = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} \subset \mathbb{C}$ is a lattice with $\Im(\tau) > 0$, $\tau = \left(\frac{\omega_2}{\omega_1}\right)$. One has en equivalence of categories

$$\begin{array}{rcl} \text{Obj} & : & \text{Elliptic curves over } \mathbb{C} \\ \text{Morphisms} & : & \text{Isogenies} \end{array} \quad \overset{\text{GAGA}}{\Leftrightarrow} \quad \begin{array}{rcc} \text{Obj} & : & \text{1-dim complex tori} \\ \text{Morphisms} & : & \begin{array}{c}\text{Non-constant holomorphic maps}\\ \text{satisfying } f(O) = O\end{array} \end{array}$$

$$\Leftrightarrow \quad \begin{array}{rcl} \text{Obj} & : & \text{Lattices } \Gamma \text{ in } \mathbb{C} \\ \text{Morphisms} & : & \operatorname{Hom}(\Gamma_1,\Gamma_2) = \{z \in \mathbb{C}^{\times} \mid z\Gamma_1 \subset \Gamma_2\} \end{array} \tag{3.8}$$

This gives $\operatorname{End}(E) \cong \{z \in \mathbb{C} \mid z\Gamma \subset \Gamma\}$ a structure of a subring of $\mathbb{C}$ and for every $z \in \operatorname{End}(E)$:

$$z = m + n\tau \quad \text{and} \quad z\tau = m' + n'\tau$$

Thus, $\operatorname{End}(E) = \mathbb{Z}$ or $\operatorname{End}(E)$ is isomorphic to an *order* in the imaginary quadratic field $\mathbb{Q}(\tau)$, i.e. a subring $\mathcal{O}$ of $\mathbb{Q}(\tau)$ that is finitely generated as a $\mathbb{Z}$-module and contains a $\mathbb{Q}$-basis of $\mathbb{Q}(\tau)$. By the Lefschetz principle, the same is true for elliptic curves over any field K of $\operatorname{char} K = 0$.

Definition 3.3.2 (CM elliptic curve)**.** Let E be an elliptic curve over $\mathbb{F}$ a subfield of $\mathbb{C}$. If $\mathrm{End}(E) \cong \mathcal{O}$ where $\mathcal{O}$ is an order of an imaginary quadratic number field K, the elliptic curve $E(\mathbb{F})$ is said to have **complex multiplication by $\mathcal{O}$**. One writes "CM" for short.

Given an elliptic curve $E(\mathbb{C})$ with complex multiplication by the maximal order $\mathcal{O}_K$, we fix an embedding $i_\infty : K \hookrightarrow \mathbb{C}$ so that every integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$ can be realised as a lattice $\Gamma_\mathfrak{a}$ in $\mathbb{C}$. One has then an isomorphism $E \cong {}^{\mathbb{C}}\!/_{\Gamma_\mathfrak{a}}$ which is fixed by $\mathcal{O}_K$. Conversely, given an integral ideal $\Gamma \triangleleft \mathcal{O}_K$, with $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\alpha \in i_\infty\left(\mathrm{End}(E_\Gamma)\right)$,

$$\alpha\omega_1 = m_1\omega_1 + m_2\omega_2 \quad \Rightarrow \quad \alpha \in \mathbb{Q}(\omega_1, \omega_2) \subset \mathbb{Q}(\tau)$$

Hence, as we have seen, α is an algebraic integer and thus, it belongs to $\mathcal{O}_K$. Hence we have the following

$$\mathrm{End}(E_\Gamma) \cong \mathcal{O}_K \Leftrightarrow \Gamma \text{ is homothetic to some integral ideal } \Gamma' \triangleleft \mathcal{O}_K$$

From the equivalence in (3.8) one has a 1-to-1 correspondence between ideal classes in $\mathrm{Cl}(K)$ and the set $\mathscr{E}ll_\mathbb{C}(K)$ of all $\mathbb{C}$-isomorphism classes of elliptic curves with complex multiplication by $\mathcal{O}_K$, given by

$$\begin{aligned} \mathrm{Cl}(K) &\xrightarrow{\sim} \mathscr{E}ll_\mathbb{C}(K) \\ [\mathfrak{a}] &\longmapsto {}^{\mathbb{C}}\!/_{\mathfrak{a}} \end{aligned} \tag{3.9}$$

For a non zero fractional ideal Γ, $\mathfrak{a}^{-1}\Gamma$ is also a fractional ideal. One thus define an action of $\mathrm{Cl}(K)$ on $\mathscr{E}ll_\mathbb{C}(K)$ (that is clearly simply transitive by (3.9)) given by

$$[a] \cdot \left[{}^{\mathbb{C}}\!/_{\Gamma}\right] := \left[{}^{\mathbb{C}}\!/_{(\mathfrak{a}^{-1}\Gamma)}\right]$$

Recall the *modular discriminant* of the lattice Γ:

$$\Delta(\Gamma) := g_2^3 - 27g_3^2 = q\prod_{n\geq 1}(1-q^n)^{24} \quad \text{with } q = e^{2\pi i\tau} \tag{3.10}$$

along with its *j-invariant* function

$$j(\Gamma) = \frac{1728 g_2^3}{g_2^3 - 27g_3^2} = \frac{(12g_2)^3}{\Delta(\Gamma)} \tag{3.11}$$

which is invariant under multiplication by $z \in \mathbb{C}^\times$. We will denote $j(E)$ (resp. $\Delta(E)$) the j invariant (resp. modular discriminant) of the lattice associated to E under complex uniformization. If one fixes an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ and considers the elliptic curve $E^\sigma(\mathbb{C})$ given by the Wierstrass equation

$$E^\sigma : Y^2 = 4X^3 - \sigma(g_2)X - \sigma(g_3),$$

then this induces a group homomorphism

$$\begin{aligned} \sigma : E(\mathbb{C}) &\longrightarrow E^\sigma(\mathbb{C}) \\ (x,y) &\longmapsto (\sigma(x), \sigma(y)) \\ O &\longmapsto O \end{aligned}$$

Lemma 3.3.3. *Let E be an elliptic curve over $\mathbb{C}$ with CM by $\mathcal{O}_K$. Then $j(E)$ is an algebraic number.*

Proof. One easily sees that

$$\mathrm{End}(E) = \mathrm{End}(E^\sigma) = \mathcal{O}_K \quad \text{and} \quad j(E^\sigma) = j(E)^\sigma$$

Thus E^σ has CM by $\mathcal{O}_K$ and by (3.9)

$$\{j(E)^\sigma \mid \sigma \in \mathrm{Aut}(\mathbb{C})\} \quad \text{is finite.}$$

Hence $j(E)$ is algebraic. □

Remark 3.3.4. One can show even more: In [ST68], J.P. Serre and J. Tate show (using Galois representations or what is more known as l-adic representation) that E has potential good reduction at all primes $\mathfrak{p}$, which implies* that $j(E)$ is integral at all primes.

We will present the main theorem of complex multiplication due to Shimura [Shi71]

Theorem 3.3.5 (Main Theorem of Complex Multiplication)**.** *Let K be an imaginary quadratic field, with ring of integers $\mathcal{O}_K$ and $E(\mathbb{C})$ be an elliptic curve with CM by $\mathcal{O}_K$. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $\sigma_{|K^{ab}} = [x, K]$ † for an idele $x \in \mathbb{J}_K$.*
Fix an analytic isomorphism $f : \mathbb{C}/_{\Gamma_\mathfrak{a}} \xrightarrow{\sim} E(\mathbb{C})$ for some fractional ideal $\mathfrak{a}$ of K, then there exists a unique complex analytic isomorphism $f' : \mathbb{C}/_{x^{-1}\Gamma} \xrightarrow{\sim} E^\sigma(\mathbb{C})$ such that the following commutes

$$\begin{tikzcd}
K/_{\mathfrak{a}} \arrow[r, "x^{-1}"] \arrow[d, "f"'] & K/_{x^{-1}\mathfrak{a}} \arrow[d, "f'"] \\
E(\mathbb{C}) \arrow[r, "\sigma"'] & E^\sigma(\mathbb{C})
\end{tikzcd}$$

The statement is also true for elliptic curves having CM with non-maximal orders, for a proof see [Shi71](Theorem 5.4) or [Lan87](1, Ch. 8, 10).

3.3.6. Hecke L-function attached to a CM elliptic curve

Let E be an elliptic curve over a field $\mathbb{F}$ with CM by $\mathcal{O}_K$ where K is a quadratic imaginary field with $K \subset \mathbb{F}$. The advantage in having complex multiplication, is that to any CM elliptic curve, one can use theorem 3.3.5 to construct a Hecke character associated it, i.e. a continuous homomorphism

$$\chi_E : \mathbb{J}_\mathbb{F} \longrightarrow \mathbb{C}^\times \quad \text{such that } \chi_E(\mathbb{F}^\times) = 1$$

with some additional properties:

*See Proposition B.3.5 in Appendix B

†This is the Artin reciprocity map $\begin{array}{ccc} \mathbb{J}_K & \longrightarrow & \mathrm{Gal}(K^{ab}/K) \\ x & \longmapsto & [x, K] \end{array}$ (see Appendix B)

Theorem 3.3.7 (Hecke character associated to a CM elliptic curve)**.** *Let E be an elliptic curve over $\mathbb{F}$ with CM by $\mathcal{O}_K$ and $K \subset \mathbb{F}$. There exists a Hecke character*

$$\chi_E : {}^{\mathbb{J}_{\mathbb{F}}}\big/_{\mathbb{F}^\times} \longrightarrow \mathbb{C}^\times$$

satisfying: For all $x = (x_v)_v \in \mathbb{J}_{\mathbb{F}}$ such that $N_{\mathbb{F}/K}(x) = y \in \mathbb{J}_K$

$$\chi_E(x)\mathcal{O}_K = \frac{y}{y_\infty}\mathcal{O}_K \subseteq \mathbb{C}$$

Proof. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $\sigma = [x, \mathbb{F}] = (x, \mathbb{F}^{ab}/\mathbb{F}) \in \mathrm{Gal}(\mathbb{F}^{ab}/\mathbb{F})$ (This is always possible because $j(E) \in \mathbb{F}$, in fact $j(E)$ generates the maximal abelian extension of K) then by the properties of the Artin reciprocity map

$$\sigma_{|K^{ab}} = (x, \mathbb{F}^{ab}/\mathbb{F})_{|K^{ab}} = (y, K^{ab}/K) = [y, K]$$

Hence, by theorem 3.3.5, one has an analytic isomorphism $f' : {}^{\mathbb{C}}\big/_{y^{-1}\Gamma} \xrightarrow{\sim} E^\sigma(\mathbb{C})$ such that the following diagrams commute:

$$\begin{tikzcd} {}^{K}\big/_{\mathfrak{a}} \arrow[r, "\cdot y^{-1}"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & {}^{K}\big/_{y^{-1}\mathfrak{a}} \arrow[d, "f'"] \\ E(\mathbb{C}) \arrow[r, "\sigma"'] & E(\mathbb{C}) \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} {}^{K}\big/_{\mathfrak{a}} \arrow[r, "\cdot \beta y^{-1}"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & {}^{K}\big/_{\mathfrak{a}} \arrow[d, "g"] \\ E(\mathbb{C}) \arrow[r, "\sigma"'] & E^\sigma(\mathbb{C}) \end{tikzcd}$$

Here, since σ fixes $\mathbb{F}$, $E^\sigma(\mathbb{C}) = E(\mathbb{C})$ and so there exists a $\beta \in K^\times$ such that $\beta y^{-1}\mathfrak{a} = \mathfrak{a}$ as $\Gamma_{\mathfrak{a}}$ and $\Gamma_{y^{-1}\mathfrak{a}}$ must be homothetic. This means:

$$\begin{tikzcd} {}^{K}\big/_{\mathfrak{a}} \arrow[r, "\cdot \alpha y^{-1}"] \arrow[d, "f"'] & {}^{K}\big/_{\mathfrak{a}} \arrow[d, "g"] \\ E(\mathbb{C}) \arrow[r, "\sigma)"'] & E(\mathbb{C}) \end{tikzcd}$$

where, say, $g = [\alpha] \circ f$. Thus, by choosing $\alpha = \varphi(\beta)$ for $[\varphi] \in \mathcal{O}_K$, one has the commutative diagram

$$\begin{tikzcd} {}^{K}\big/_{\mathfrak{a}} \arrow[r, "\cdot \alpha y^{-1}"] \arrow[d, "f"'] & {}^{K}\big/_{\mathfrak{a}} \arrow[d, "f''"] \\ E(\mathbb{F}^{ab}) \arrow[r, "(x, \mathbb{F}^{ab}/\mathbb{F})"'] & E(\mathbb{F}^{ab}) \end{tikzcd} \quad \text{and} \quad \alpha\mathcal{O}_K = (y) \lhd K \tag{3.12}$$

Remark 3.3.8. (i) Note that such an α always exists and is unique, since otherwise, for another such $\alpha' \in K^\times$

$$\begin{tikzcd} {}^{K}\big/_{\mathfrak{a}} \arrow[rr, equal] & & {}^{K}\big/_{\mathfrak{a}} \\ & {}^{K}\big/_{\mathfrak{a}} \arrow[ul, "\cdot \alpha' y^{-1}"] \arrow[ur, "\cdot \alpha y^{-1}"'] & \end{tikzcd} \quad \Rightarrow \quad \cdot\alpha\alpha' = \mathrm{id}_{{}^{K}/_{\mathfrak{a}}} \quad \Rightarrow \quad \alpha = \alpha'$$

(ii) $\alpha = \alpha_x$ depends on x but not on the isomorphism $f : \mathbb{C}/\Gamma_{\mathfrak{a}} \xrightarrow{\sim} E(\mathbb{C})$. Indeed, for any other choice $g : \mathbb{C}/\Gamma'_{\mathfrak{a}} \xrightarrow{\sim} E(\mathbb{C})$, there exists a $\beta' \in K^\times$ such that $\beta'\mathfrak{a}' = \mathfrak{a}$ and $g \circ f^{-1} \in \mathrm{Aut}(E)$. This implies that $g(\cdot) = f(u\beta'\cdot)$ for $u \in \mathcal{O}_K^\times$. Thus, from the diagram (3.12), one has

$$\begin{tikzcd}
K/\mathfrak{a} \arrow[r, "\cdot\alpha y^{-1}"] \arrow[d, "g"'] & K/\mathfrak{a} \arrow[d, "g"] \\
E(\mathbb{F}^{ab}) \arrow[r, dashed, "(x,\mathbb{F}^{ab}/\mathbb{F})"'] \arrow[d, "f(u\beta'\cdot)"'] & E(\mathbb{F}^{ab}) \arrow[d, "f(u\beta'\cdot)"] \\
E(\mathbb{F}^{ab}) \arrow[r, "(x,\mathbb{F}^{ab}/\mathbb{F})"'] & E(\mathbb{F}^{ab})
\end{tikzcd}$$

and for all $z \in K/\mathfrak{a}$:

$$g(z)^{(x,\mathbb{F}^{ab}/\mathbb{F})} = f(u\beta' z)^{(x,\mathbb{F}^{ab}/\mathbb{F})} \underset{\substack{\uparrow \\ \text{by theorem 3.3.5}}}{=} f(\alpha y^{-1} u\beta' z) = g(\alpha y^{-1} z)$$

Now, define

$$\begin{aligned}
\chi_E : \mathbb{J}_{\mathbb{F}} &\longrightarrow \mathbb{C}^\times \\
x = (x_v)_v &\longmapsto y_\infty^{-1}\alpha_x
\end{aligned}$$

where $y_\infty \in \mathbb{C}^\times$ is the component of $y \in \mathbb{J}_K$ corresponding to the unique archimedean place on K.

- χ_E is a group homomorphism: For $x, x' \in \mathbb{J}_{\mathbb{F}}$ and by uniqueness in the above remark, one sees that
$$\alpha_{(xx')} = \alpha_x \alpha_{x'}$$
- $\chi_E(\mathbb{F}^\times) = 1$: For any β coming from $\mathbb{F}^\times$, $(x, \mathbb{F}^{ab}/\mathbb{F}) = 1$ for the idèle $x \in \mathbb{J}_{\mathbb{F}}$ with $\alpha_x = \varphi(\beta)$ for $[\varphi] \in \mathcal{O}_K$. Hence, by (3.12) once again,
$$\alpha_x \mathcal{O}_K = (y)\mathcal{O}_K = N_{\mathbb{F}/K}(\beta)\mathcal{O}_K$$
one finally gets that $\alpha_x = N_{\mathbb{F}/K}(\beta)$ and clearly
$$N_{\mathbb{F}/K}(\beta)_\infty = \prod_{\substack{i_\infty : \mathbb{F} \hookrightarrow \mathbb{C} \\ i_{\infty|K} = 1}} \beta^{i_\infty} = N_{\mathbb{F}/K}(\beta)$$
- χ_E is continuous: The proof uses the same argument as in the proof of (3.1.18) using remark 3.1.13.

$\square$

Hence, one has a Hecke L-function attached to the Hecke character χ_E. One relates this Hecke L-function to the global L-function of a CM elliptic curve through Deuring's theorem:

Theorem 3.3.9 (Deuring). *Let E be an elliptic curve over $\mathbb{F}$ with CM by $\mathcal{O}_K$ and suppose that $K \subset \mathbb{F}$. Then the global L-series of $E(\mathbb{F})$ is given by*

$$L(E(\mathbb{F}), s) = L(s, \chi_E)L(s, \overline{\chi}_E)$$

(See (B.3.10) in appendix B for a sketch of the proof).

Now, combining this with theorem 3.2.10, we show that the L-function of a CM elliptic curve has an analytic continuation and satisfies a functional equation as follow:

Corollary 3.3.10. *Let E be an elliptic curve over $\mathbb{F}$ with CM by OK, $K \subset \mathbb{F}$. Then the L-function of $E(\mathbb{F})$ has an analytic continuation to the entire complex plane and satisfies the functional equation*

$$\Lambda(E(\mathbb{F}), s) = \epsilon(s, \chi)\Lambda(E(\mathbb{F}), 2 - s)$$

Where $\epsilon(s, \chi) = \pm 1$, $\mathfrak{f}$ the conductor of χ_E, $|\Delta_{\mathbb{F}}|$ the absolute discriminant of $\mathbb{F}$ and

$$\Lambda(E(\mathbb{F}), s) = L(E(\mathbb{F}), s) \cdot \left(N_{\mathbb{F}/\mathbb{Q}}(\mathfrak{f})|\Delta_{\mathbb{F}}|\right)^s (2\pi)^{-s[\mathbb{F}:\mathbb{Q}]}\Gamma(s)^{[\mathbb{F}:\mathbb{Q}]}$$

Proof. Let E be an elliptic curve over $\mathbb{F}$ with Hecke characters χ_E. Clearly, χ_E and $\overline{\chi_E}$ have the same conductor $\mathfrak{f}$ and only $\frac{[\mathbb{F}:\mathbb{Q}]}{2}$ complex embeddings. Recall that for an ideal $\mathfrak{a} \in I_{\mathfrak{f}}$:

$$\begin{aligned}
&\chi_E(\mathfrak{a})\overline{\chi_E(\mathfrak{a})} = N_{K/\mathbb{Q}}(\chi_E(\mathfrak{a})) = N_{\mathbb{F}/\mathbb{Q}}(\mathfrak{a}) \quad \Rightarrow \quad \overline{\chi_E(\mathfrak{a})} = \chi_E(\mathfrak{a})^{-1}N_{\mathbb{F}/\mathbb{Q}}(\mathfrak{a}) \\
\Rightarrow \quad & L(s, \overline{\chi_E}) = \sum_{\substack{0 \neq \mathfrak{a} \triangleleft \mathcal{O}_K \\ (\mathfrak{a},\mathfrak{f})=1}} \frac{\overline{\chi_E}(\mathfrak{a})}{N_{\mathbb{F}/\mathbb{Q}}(\mathfrak{a})^s} = \sum_{\substack{0 \neq \mathfrak{a} \triangleleft \mathcal{O}_K \\ (\mathfrak{a},\mathfrak{f})=1}} \frac{\chi_E^{-1}(\mathfrak{a})}{N_{\mathbb{F}/\mathbb{Q}}(\mathfrak{a})^{s-1}} = L(s-1, \chi_E^{-1})
\end{aligned}$$

Thus, one has by definition B.3.8 and cB.3.10:

$$\begin{aligned}
\Lambda(E(\mathbb{F}), s) &= \left(N_{\mathbb{F}/\mathbb{Q}}(\mathfrak{f})|\Delta_{\mathbb{F}}|\right)^s (2\pi)^{-s[\mathbb{F}:\mathbb{Q}]}\Gamma(s)^{[\mathbb{F}:\mathbb{Q}]}L(s, \chi_E)L(s-1, \chi_E^{-1}) \\
&= \left(N_{\mathbb{F}/\mathbb{Q}}(\mathfrak{f})|\Delta_{\mathbb{F}}|\right)^{\frac{1}{2}} \left(\frac{\Gamma(s)}{8\pi\Gamma(s-1)}\right)^{\frac{[\mathbb{F}:\mathbb{Q}]}{2}} \Lambda(s, \chi_E)\Lambda(s-1, \chi_E^{-1}) \\
&= (-1)^{\frac{[\mathbb{F}:\mathbb{Q}]}{2}}\epsilon(\chi_E)\epsilon(\chi_E^{-1})\left(N_{\mathbb{F}/\mathbb{Q}}(\mathfrak{f})|\Delta_{\mathbb{F}}|\right)^{\frac{1}{2}} \left(\frac{\Gamma(s)}{8\pi\Gamma(s-1)}\right)^{\frac{[\mathbb{F}:\mathbb{Q}]}{2}} \Lambda(2-s, \chi_E)\Lambda(s-1, \chi_E^{-1}) \\
&= \epsilon(\chi_E)\left(N_{\mathbb{F}/\mathbb{Q}}(\mathfrak{f})|\Delta_{\mathbb{F}}|\right)^{2-s} (2\pi)^{(2-s)[\mathbb{F}:\mathbb{Q}]}\Gamma(2-s)^{[\mathbb{F}:\mathbb{Q}]}L(s, \chi_E)L(s-1, \chi_E^{-1}) \\
&= \epsilon(\chi_E)\Lambda(E(\mathbb{F}), 2-s).
\end{aligned}$$

Where $\epsilon(\chi_E) = (-1)^{\frac{[\mathbb{F}:\mathbb{Q}]}{2}} \prod_v \epsilon_v(\chi_v, s)\epsilon_v(\chi_v^{-1}, s)$ is the global epsilon factor in theorem 3.2.10, which has norm 1. Although, calculations show that

$$\overline{\epsilon(\chi_E)} = \epsilon(\chi_E) \quad \Rightarrow \quad \epsilon(\chi_E) = \pm 1$$

□

Remark 3.3.11. More generally, the result is also true when the field of definition of $E(\mathbb{F})$ does not contain the CM field K. In this case:

$$\Lambda(E(\mathbb{F}), s) = L(E(\mathbb{F}), s) \cdot \left(N_{\mathbb{F}'/\mathbb{Q}}(\mathfrak{f}')|\Delta'_{\mathbb{F}}|\right)^s (2\pi)^{-s[\mathbb{F}:\mathbb{Q}]}\Gamma(s)^{[\mathbb{F}:\mathbb{Q}]}$$

where $\mathbb{F}' := \mathbb{F}K$ and $\mathfrak{f}'$ is the conductor of the Hecke character $\chi'_E : \mathbb{J}_{\mathbb{F}'}/(\mathbb{F}')^\times \longrightarrow \mathbb{C}^\times$ attached to $E(\mathbb{F}')$.

3.3.12. Conjectures related to special values of L-function and further discussion

We finish this section by highlighting the importance of the study of special values of L-functions associated to elliptic curves. One of the main conjectures about this topic is known as the Hasse-Weil conjecture:

Conjecture 3.3.13 (Hasse-Weil)**.** *Let E be an elliptic curve over a number field K. The L-function of $E(K)$ has an analytic continuation to the entire complex plane, and satisfies the functional equation*

$$\Lambda(E, s) = \epsilon \cdot \Lambda(E, 2 - s)$$

with $\epsilon = \pm 1$, $\mathfrak{f}$ the conductor of χ_E and

$$\Lambda(E, s) = L(E, s) \cdot \left(N_{K/\mathbb{Q}}(\mathfrak{f})|\Delta_{\mathbb{F}}|\right)^{\frac{s}{2}} (2\pi)^{-s[K:\mathbb{Q}]}\Gamma(s)^{[K:\mathbb{Q}]}$$

This conjecture has been verified for the class of CM elliptic curves as seen in corollary 3.3.10. This leads to a whole set of conjectural properties known as the Birch and Swinneton-Dyer conjectures. Roughly, these conjectures describe somehow, the set of rational solutions to equations defining an elliptic curve. The most famous formulation of the Birch and Swinneton-Dyer conjecture links the arithmetic of the elliptic curve E over a number field K, to the behaviour of its L-function at $s = 1$:

Conjecture 3.3.14 (Birch and Swinneton-Dyer - 1st)**.** *Let E be an elliptic curve over a number field K, with L-function having an analytic continuation to the complex plane. Let $r_{E(K)}$ denote the rank of $E(K)$. Then $r_{E(K)}$ is equal to the order of the zero of $L(E(K), s)$ at the point $s = 1$.*

Wiles and Coates showed in [CW77] that in the case where the elliptic curve had CM, one has the following:

Theorem 3.3.15. *Let E be a CM elliptic curve over a number field $K \subset \mathbb{F}$. Suppose $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = K$. If $r_{E(K)} \neq 0$, then*

$$L(E(\mathbb{F}), 1) = 0$$

This result has been extended to fields $\mathbb{F}$ which are abelian extensions of K with some particular assumptions (see [Art78]) and K. Rubin has given in [Rub81] some precise information about the order of vanishing of $L(E(\mathbb{F}), s)$ at $s = 1$.

4

Poincaré bundle and reduced theta functions

The theory of theta function appears as a very important part of the study of Abelian varieties. In modern terminology, these functions appear as holomorphic sections of line bundles over an Abelian variety. We introduce the general theory of theta functions and show how; in the case of an elliptic curve; it is related to the Eisenstein-Kronecker numbers $e^*_{a,b}$.

4.1. Review of the classical theory of theta functions

Let V be a g-dimensional complex vector space, and $\Gamma \cong \mathbb{Z}^{2g}$ a lattice in V. A *theta function* is a holomorphic function on V which is quasi-periodic (recall that the only holomorphic Γ-periodic functions are the constant ones) with respect to Γ i.e. for all $\gamma \in \Gamma$, there exists a holomorphic map $f_\gamma : V \to \mathbb{C}$ such that

- $\theta(z+\gamma) = f_\gamma(z)\theta(z)$,
- f_γ is a factor of automorphy.

In a more modern terminology, this reads into following:

4.1.1. Meromorphic sections of a line bundle

Consider the complex torus $\mathbb{T} := {}^{\mathbb{C}}\!/_{\Gamma}$. It is a complex Lie group of dimension g with a holomorphic covering map $\pi : V \to \mathbb{T}$. Moreover, the complex structure on V induces the decomposition

$$\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = V^* \oplus \overline{V}^*$$

where

$$V^* := \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) = \{\mathbb{C}\text{-linear forms } l\} \quad \text{and} \quad \overline{V}^* := \mathrm{Hom}_{\mathbb{C}}(\overline{V},\mathbb{C}) = \{\mathbb{C}\text{-anti-linear forms } \bar{l}\}$$

This decomposition induces the decomposition

$$H^1(\mathbb{T},\mathbb{C}) \cong H^{1,0}(\mathbb{T}) \oplus H^{0,1}(\mathbb{T})$$

where $H^{1,0}(\mathbb{T})$ is the subspace spanned by the classes of $dz_1, \dots, dz_g$ and $H^{0,1}(\mathbb{T})$ is the one spanned by the classes of $d\overline{z}_1, \dots, d\overline{z}_g$. The isomoprhism, in this case, is given by

$$\begin{aligned} \delta^1 : \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) &\xrightarrow{\sim} H^1(\mathbb{T}, \mathbb{C}) \\ l &\longmapsto [dl] \end{aligned}$$

Recall that the Picard group $\mathrm{Pic}(X)$ of a complex manifold X is the group of isomoprhism classes of holomorphic line bundles on X. It is canonically isomorphic to $H^1(X, \mathcal{O}_X^\times)$. One has an exact sequence of sheaves, commonly known as the *exponential exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_X & \longrightarrow & \mathcal{O}_X & \xrightarrow{\exp} & \mathcal{O}_X^\times & \longrightarrow & 1 \\ & & & & f & \longmapsto & \exp(2\pi i f) & & \end{array}$$

This gives rise to a long exact sequence in cohomology

$$H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X).$$

In particular, for $X = V (\cong \mathbb{C}^g)$, by Dolbeault's theorem and the Poincaré $\overline{\partial}$-lemma (See [PG78] Dolbeaut Theorem p.45 - Computation 2. p.46), one sees that

$$H^1(V, \mathcal{O}_V) \cong 0 \quad \text{and} \quad H^2(V, \mathbb{Z}) \cong 0.$$

Thus, $\mathrm{Pic}(V)$ is trivial. Now consider a line bundle $\mathcal{L}$ on $\mathbb{T}$, then one has the following diagram

$$\begin{tikzcd} \pi^*\mathcal{L} = V \times_{\mathbb{T}} \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\ V \arrow[r, "\pi"] & \mathbb{T} \end{tikzcd}$$

One lifts the group action of Γ on V to an action of $\pi^*\mathcal{L}$. Since $\pi^*\mathcal{L}$ is trivial as a line bundle on V, by choosing a trivialization $\pi^*\mathcal{L} \xrightarrow{\sim} V \times \mathbb{C}$, one gets an action of Γ on $V \times \mathbb{C}$ and every line bundle $\mathcal{L}$ on $\mathbb{T}$ is the quotient of $V \times \mathbb{C}$ by this action.

Now, every $\gamma \in \Gamma$ acts linearly on the fibres, i.e

$$\gamma \cdot (\xi, v) = (v + \xi, e_\gamma(\xi) v) \quad \text{for} \quad \gamma \in \Gamma,\ v \in V,\ \xi \in \mathbb{C}$$

where e_γ is a holomorphic invertible function on V. Thus, for it to be a group action, one needs the following *cocycle condition**:

$$e_{\gamma_1 + \gamma_2}(v) = e_{\gamma_1}(v + \gamma_2) e_{\gamma_2}(v) \tag{4.1}$$

*From a Group Cohomology point of view, this is the cocycle condition for the Čech 1-cocycle map
$$\begin{array}{ccc} \Gamma & \longrightarrow & H^0(V, \mathcal{O}_V^*) \\ \gamma & \longmapsto & e_\gamma \end{array}$$

Such functions e_γ are called *multipliers* or *multiplicators* and a family $(e_\gamma)_\gamma$ defines a line bundle $\mathcal{L}$ on $\mathbb{T}$.

Hence, given a family of multipliers $(e_\gamma)_\gamma$, one has a corresponding theta function $\theta : V \longrightarrow \mathbb{C}$ defined by

$$\theta(v+\gamma) = e_\gamma(v)\theta(z) \quad \text{ for all } \gamma \in \Gamma, v \in V$$

Moreover, the space $H^0(\mathbb{T}, \mathcal{L})$ of global holomorphic sections of V can be identified with the space of theta functions characterised by a family of multipliers $(e_\gamma)_\gamma$ as follow:

- A section s of the line bundle $V \times_{\mathbb{T}} \mathcal{L}$ is of the form $v \longmapsto (v, \theta(v))$ where $\theta : V \longrightarrow \mathbb{C}$ is holomorphic. It is invariant under the action of Γ if and only if θ is a theta function for $(e_\gamma)_\gamma$. Now a global section s of $\mathcal{L}$ lifts to a section $\widehat{s} := \pi^* s$ of $V \times_{\mathbb{T}} \mathcal{L}$ defined by $z \longmapsto (z, s \circ \pi(z))$ and one has

$$\widehat{s}(v+\gamma) = (v+\gamma, s \circ \pi(v)) = \gamma \cdot \widehat{s}(v+\gamma)$$

- On the other hand, every section that is Γ-invariant MUST come from a section of $\mathcal{L}$.

We end this section by the following remark:

Remark 4.1.2. The line bundle $\mathcal{L} \otimes \mathcal{L}'$ is defined by the family of multipliers $(e_\gamma e'_\gamma)_{\gamma\in\Gamma}$ where $\mathcal{L}, \mathcal{L}'$ are defined by the families of multipliers $(e_\gamma, e'_\gamma)_{\gamma\in\Gamma}$, respectively. (observe that this is nothing but the quotient $V \times (\mathbb{C}\otimes\mathbb{C})$ by the action $\gamma \cdot (v, \xi \otimes \xi') = (v+\gamma, e_\gamma e'_\gamma \xi \otimes \xi'))$. This induces a surjective group homomorphism

$$H^0(V, \mathscr{O}_V^*) \longrightarrow\!\!\!\!\!\rightarrow \operatorname{Pic}(\mathbb{T})$$

whose kernel consists of *coboundaries*† i.e elements $e_\gamma(v) = \frac{h(v+\gamma)}{h(v)}$ where $h : V \longrightarrow \mathbb{C}^\times$ is holomorphic (this comes from the fact that such elements must define a line bundle that admits non-everywhere vanishing sections).

4.1.3. Reduced Theta functions

The next step would be to construct a canonical family of multipliers that would uniquely determine a line bundle, and thus the corresponding theta function. This will involve Hermitian forms on V. Recall that a *hermitian* form H on V is an $\mathbb{R}$-bilinear form which is $\mathbb{C}$-antilinear in the first argument and $\mathbb{C}$-linear in the second. We will write $H(x,y) = S(x,y) + iE(x,y)$, where S is a symmetric $\mathbb{R}$-linear form on V and E a skew-symmetric one. Moreover one has that

† This is nothing but the coboundary condition for the 1-cocycle e_γ. Moreover, one has a group isomorphism $H^1\big(\Gamma, H^0(V, \mathscr{O}_V^*)\big) \cong \operatorname{Pic}(\mathbb{T})$

- $E(x,y) = E(ix,iy)$, $S(x,y) = E(x,iy)$ and there is a 1-to-1 correspondence

$$\mathrm{Her}(V) := \left\{ \begin{array}{c} \text{Hermitian forms} \\ H : V \times V \longrightarrow \mathbb{C} \end{array} \right\} \quad \longleftrightarrow \quad \mathrm{Alt}^2(V,\mathbb{R}) := \left\{ \begin{array}{c} \text{Alternating bilinear forms} \\ E : V \times V \longrightarrow \mathbb{R} \end{array} \right\}$$

- H is non-degenerate $\Leftrightarrow$ E is non-degenerate $\Leftrightarrow$ S is non-degenerate.

From now on, we suppose that H is a hermitian form such that $E(\Gamma \times \Gamma) \subseteq \mathbb{Z}$. A *semi-character* χ associated to a hermitian form H is a map $\chi : \Gamma \longrightarrow \mathbb{S}^1$ satisfying:

$$\chi(\gamma_1 + \gamma_2) = \exp(\pi i E(\gamma_1,\gamma_2))\chi(\gamma_1)\chi(\gamma_2) \quad \text{for all } \gamma_1,\gamma_2 \in \Gamma.$$

Note that by definition, unitary characters on Γ are semi-characters associated to the hermitian form $H = 0$. We denote by $\mathscr{P}(\Gamma)$ the set of pairs (H,χ) (it actually is a group under the law $(H_1,\chi_1)\cdot(H_2,\chi_2) = (H_1 + H_2, \chi_1\chi_2)$).
Pose

$$e_\gamma(v) = \chi(\gamma)\exp\left(\pi H(v,\gamma) + \frac{\pi}{2}H(\gamma,\gamma)\right)$$

Observe that, for $\gamma_1,\gamma_2 \in \Gamma$ and $v \in V$:

$$\begin{aligned} e_{\gamma_1+\gamma_2}(v) &= \chi(\gamma_1)\chi(\gamma_2)\exp\left(\pi i E(v,\gamma) + \pi H(v,\gamma_1+\gamma_2) + \frac{\pi}{2}H(\gamma_1+\gamma_2,\gamma_1+\gamma_2)\right) \\ &= \chi(\gamma_1)\exp\left(\pi H(v+\gamma_2,\gamma_1) + \frac{\pi}{2}H(\gamma_1,\gamma_1)\right)\cdot\chi(\gamma_2)\exp\left(\pi H(v,\gamma_2) + \frac{\pi}{2}H(\gamma_2,\gamma_2)\right) \\ &= e_{\gamma_1}(v+\gamma_2)e_{\gamma_2}(v) \end{aligned}$$

Thus, $(e_\gamma)_{\gamma\in\Gamma}$ is a family of multipliers since e_γ satisfies the cocycle condition (4.1). Each pair (H,χ) determine (through $(e_\gamma)_{\gamma\in\Gamma}$) a line bundle that will be denoted by $\mathcal{L}(H,\chi)$ and one has a group homomorphism

$$\begin{aligned} \mathscr{P}(\Gamma) &\longrightarrow \mathrm{Pic}(\mathbb{T}) \\ (H,\chi) &\longmapsto \mathcal{L}(H,\chi) \end{aligned}$$

by noticing that

$$\begin{aligned} \mathcal{L}(H_1,\chi_1)\otimes\mathcal{L}(H_2,\chi_2) &\cong \mathcal{L}(H_1+H_2,\chi_1\chi_2) \\ \mathrm{inv}^*\mathcal{L}(H,\chi) &= \mathcal{L}(H,\chi^{-1}) \quad \text{where } \mathrm{inv} : v \in \mathbb{T} \longmapsto -v \in \mathbb{T}. \end{aligned}$$

Appell-Humbert theorem shows that it is an isomorphism ([DM70], Chapter 1, Section 2, p.20), which classifies all isomorphism classes of holomorphic line bundles on $\mathbb{T}$.

Theorem 4.1.4 (Appell-Humbert)**.** *There is a canonical isomorphism of exact sequences*

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Hom}(\Gamma,\mathbb{S}^1) \arrow[r] \arrow[d, "\wr"] & \mathscr{P}(\Gamma) \arrow[r] \arrow[d, "\wr"] & \{H \in \mathrm{Her}(V) \mid E(\Gamma\times\Gamma) \subseteq \mathbb{Z}\} \arrow[r] \arrow[d, "\wr"] & 0 \\ 1 \arrow[r] & \mathrm{Pic}^0(\mathbb{T}) \arrow[r] & \mathrm{Pic}(\mathbb{T}) \arrow[r] & \mathrm{Im}\left(H^1(X,\mathscr{O}_X^\times) \xrightarrow{c_1} H^2(X,\mathbb{Z})\right) \arrow[r] & 0 \end{tikzcd}$$

Remark 4.1.5 (Chern class). For a line bundle $\mathcal{L}(H,\chi) \in \mathrm{Pic}(\mathbb{T})$ the image $c_1(\mathcal{L})$ under the map

$$c_1 : H^1(X, \mathscr{O}_X^\times) \longrightarrow H^2(X, \mathbb{Z})$$

is called the *first Chern class* of $\mathcal{L}(H,\chi)$. It can be associated to a (unique) alternating form $E \in \mathrm{Alt}^2(\Gamma, \mathbb{Z})$ (or a Hermitian form $H \in \mathrm{Her}(\mathbb{T})$). Hence, Chern classes of line bundles on $\mathbb{T}$ can be identified with hermitian forms on V satisfying $E(\Gamma,\Gamma) \subseteq \mathbb{Z}$.

An important consequence of Ampell-Humbert theorem is the following: Consider the translation map

$$\begin{aligned} t_v : \mathbb{T} &\longrightarrow \mathbb{T} \\ z &\longmapsto z + v \end{aligned}$$

By pulling back along π, one gets a map

$$\begin{tikzcd} t^*_{\pi(v)}\mathcal{L} \cong \mathcal{L}' \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\ V \arrow[r, "\pi"] & \mathbb{T} \end{tikzcd}$$

where, clearly, $\mathcal{L}'$ is defined by the multipliers $(t^*_{\pi(v)} e_\gamma)_{\gamma\in\Gamma}$. However, direct calculations show that for $v \in V$:

$$t^*_{\pi(v)} e_\gamma(v') = e_\gamma(v'+v) = e_\gamma(v') \exp\left(\pi H(v,\gamma)\right) \tag{4.2}$$

By remark 4.1.2, one can multiply both sides by a coboundary $\frac{h(v+\gamma)}{h(v)}$ without changing the associated line bundle $t^*_{\pi(v)}\mathcal{L}$. Thus, for $h : u \longmapsto \exp\left(-\pi H(v,u)\right)$:

$$\begin{aligned} t^*_{\pi(v)} e_\gamma(v') \frac{h(v+\gamma)}{h(v)} &= \chi(\gamma) \exp\left(\pi H(v',\gamma) + \frac{\pi}{2} H(\gamma,\gamma)\right) \exp\left(2\pi i \Im\, H(v,\gamma)\right) \\ &= \left[\chi(\gamma) \underbrace{\exp\left(2\pi i E(v,\gamma)\right)}_{:=\alpha_v(\gamma)}\right] \exp\left(\pi H(v',\gamma) + \frac{\pi}{2} H(\gamma,\gamma)\right). \end{aligned}$$

Define the character

$$\begin{aligned} \alpha_v : \ \Gamma &\longrightarrow \mathbb{S}^1 \\ \gamma &\longmapsto \exp\left(2\pi i E(v,\gamma)\right) \end{aligned}$$

Then, one gets

$$t^*_{\pi(v)} \mathcal{L}(H,\chi) \cong \mathcal{L}(H,\chi') \qquad \text{where } \chi'(\gamma) = \chi(\gamma)\alpha_v(\gamma) \tag{4.3}$$

This implies, in particular, the following:

Theorem 4.1.6 (theorem of the square)**.** *For all $v, w \in V$ and line bundle $\mathcal{L}$ on $\mathbb{T}$*

$$t^*_{\pi(v)+\pi(w)}\mathcal{L} \otimes \mathcal{L} = t^*_{\pi(v)}\mathcal{L} \otimes t^*_{\pi(w)}\mathcal{L}.$$

Moreover, one has a group homomorphism

$$\begin{aligned} \phi_{\mathcal{L}} : \mathbb{T} &\longrightarrow \mathrm{Pic}^0(\mathbb{T}) \\ z &\longmapsto t^*_z\mathcal{L} \otimes \mathcal{L}^{-1} \\ 0 &\longmapsto t^*_0\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathscr{O}_{\mathbb{T}} \end{aligned}$$

with

$$K_{\mathcal{L}} := \ker(\phi_{\mathcal{L}}) = \{z \in \mathbb{T} \mid t^*_z\mathcal{L} \cong \mathcal{L}\}$$

Remark 4.1.7. Note that the morphism $\phi_{\mathcal{L}}$ factors through

$$\begin{array}{rclcl} \mathbb{T} & \xrightarrow{\alpha} & \mathrm{Hom}(\Gamma, \mathbb{S}^1) & \xrightarrow{\sim} & \mathrm{Pic}^0(\mathbb{T}) \\ \pi(v) = z & \longmapsto & (\chi_0 : \gamma \to \alpha_v(\gamma)) & \longmapsto & \mathcal{L}(0, \chi_0) \end{array}$$

Since Γ is a free $\mathbb{Z}$-module, one has a group homomorphism $\varphi : \Gamma \longrightarrow \mathbb{R}$ such that for any $\chi \in \mathrm{Hom}(\Gamma, \mathbb{S}^1)$:

$$\chi(\gamma) = \alpha_v \circ \varphi(\gamma)$$

Moreover,

(i) If E is non-degenerate, then $\phi_{\mathcal{L}}$ is surjective: In fact, by non-degeneracy, there exists an element $v \in V$ such that $\varphi(z) = E(z, v)$ (by $\mathbb{R}$-linear extension) and thus, $\varphi \circ \pi = \chi$. This shows that in the case of a non-degenerate line bundle $\mathcal{L}$, $\phi_{\mathcal{L}}$ is surjective and has finite kernel (as E induces an $\mathbb{R}$-linear isomorphism $V \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ which identifies -as one will see later- $\{v \in V | E(v, \gamma) \in \mathbb{Z} \quad \forall \gamma \in \Gamma\}$ with the dual lattice $\Gamma^{\vee}$).

(ii) If E is unimodular, then $\phi_{\mathcal{L}}$ is an isomorphism: In the next section, one sees that $\mathrm{Pic}^0(\mathbb{T})$ can be endowed with a complex structure, as a complex torus. One shows that in this case, $\phi_{\mathcal{L}}$ is an isomorphism of complex tori.

Definition 4.1.8 (reduced theta function)**.** A theta function ϑ, associated to a meromorphic section of a line bundle of the form $\mathcal{L}(H, \chi)$, is called a **reduced** or **canonical** theta function. In other words, a **reduced** theta function ϑ is a meromorphic function on V satisfying for $v \in V$ and $\gamma \in \Gamma$:

$$\vartheta(v + \gamma) = \chi(\gamma) \exp\left(\pi H(v, \gamma) + \frac{\pi}{2} H(\gamma, \gamma)\right) \vartheta(v) \tag{4.4}$$

Remark 4.1.9 (Meromorphic sections and divisors). Recall that on a compact manifold, a meromorphic function is uniquely determined, up to a constant, by its divisors. In our setting, given a divisor $D \in \mathrm{Div}(\mathbb{T})$, there is an associated line bundle $\mathscr{O}_{\mathbb{T}}(D) = [D]$ whose sections on some open U are meromorphic functions $(f) + D_{|U}$. Conversely, defining a meromorphic section s of a line bundle $\mathcal{L}$ (through local trivializations), s has an associated (and well defined) divisor $D = (s)$ with $\mathcal{L} \cong \mathscr{O}_{\mathbb{T}}(D)$. Hence, by choosing $\mathcal{L} = \mathcal{L}(H, \chi)$, one associates to any divisor $D \in \mathrm{Div}(\mathbb{T})$, a unique reduced theta function ϑ_D; determined up to a constant. In §4.2.5 an example of a reduced theta function, corresponding to the divisor $D = [0]$ will be constructed.

4.1.10. Translation of reduced theta functions

Let $\mathcal{L}(H, \chi)$ be a line bundle on $\mathbb{T}$. one has from (4.3), that

$$t^*_{z=\pi(v)}\mathcal{L}(H, \chi) \cong \mathcal{L}(H, \chi \cdot \alpha_v)$$

Thus, any meromorphic section s_D of $\mathcal{L}(H, \chi)$ induces a reduced theta function ϑ_D on $\mathcal{L}(H, \chi \cdot \alpha_v)$, corresponding to a meromorphic section $t^*_z s_D$. Extend the semi-character χ into a map $\widetilde{\chi} : V \longrightarrow \mathbb{C}^\times$ and define

$$\begin{aligned} e\ : V \times V &\longrightarrow \mathbb{C}^\times \\ (v, w) &\longmapsto e_w(v) := \widetilde{\chi}(w) \exp\left(\pi H(v, w) + \frac{\pi}{2} H(w, w)\right) \end{aligned} \tag{4.5}$$

Clearly, one sees that the restriction $e_{|V\times\Gamma}$ defines a system of multipliers $(e_\gamma)_{\gamma\in\Gamma}$ for ϑ_D. Since $\vartheta_D(v+\gamma) = e_\gamma(v)\vartheta_D(v)$, it is only natural to define the *translation* of ϑ_D to be the meromorphic function

$$\vartheta_D^{+w}(v) := e_w(v)^{-1}\vartheta_D(v + w). \tag{4.6}$$

One first observes that for $\gamma \in \Gamma$

$$\begin{aligned} \vartheta_D^{+w}(v + \gamma) &= e_w(v + \gamma)^{-1}\vartheta_D(v + w + \gamma) \\ &= \exp\left[-\pi H(\gamma, w)\right] e_\gamma(v + w) e_w(v)^{-1}\vartheta_D(v + w) \\ &= \chi(\gamma) \exp\left[\pi H(w, \gamma) - \pi H(\gamma, w)\right] \exp\left[\pi H(v, \gamma) + \frac{\pi}{2} H(\gamma, \gamma)\right] \vartheta_D^{+w}(v) \\ &= \chi(\gamma) \exp\left[2\pi i E(w, \gamma)\right] \exp\left[\pi H(v, \gamma) + \frac{\pi}{2} H(\gamma, \gamma)\right] \vartheta_D^{+w}(v). \end{aligned}$$

Thus, by (4.2), ϑ_D^{+w} is indeed a reduced theta function for $t^*_z\mathcal{L}(H, \chi)$; associated to the section $t^*_z s$. Although the family $(e_w)_{w\in\mathbb{C}}$ is clearly not a multiplier for $w \in \mathbb{C} \smallsetminus \Gamma$, it nevertheless satisfies the following properties:

Proposition 4.1.11. *Let $w_1, w_2 \in V$ and χ be a semi character for $H \in \mathrm{Her}(V)$. Let*

$$\chi(w_1, w_2) := \widetilde{\chi}(w_1 + w_2)\widetilde{\chi}(w_1)^{-1}\widetilde{\chi}(w_2)^{-1} \exp\left(2\pi i E(w_1, w_2)\right)$$

Then, one has the following properties:

(i) $e_{w_1+w_2}(v) = \chi(w_1, w_2) e_{w_1}(v + w_2) e_{w_2}(v)$.

(ii) $\vartheta_D^{+w_1}(v+w_2) = \chi(w_1,w_2)e_{w_1}(v)\vartheta_D^{+(w_1+w_2)}(v)$.

(iii) $(\vartheta_D^{+w_1})^{+w_2} = \chi(w_1,w_2)\vartheta_D^{+(w_1+w_2)}$.

(iv) $(\vartheta_D^{+w_1})^{+w_2} = \exp\left(2\pi i E(w_1,w_2)\right)(\vartheta_D^{+w_2})^{+w_1}$.

Proof. (i) From (4.5) one has

$$\begin{aligned}
e_{w_1+w_2}(v) &= \widetilde{\chi}(w_1+w_2)\exp\left(\pi H(v,w_1+w_2)+\frac{\pi}{2}H(w_1+w_2,w_1+w_2)\right)\\
&= \chi(w_1,w_2)\widetilde{\chi}(w_2)\exp\left(\pi H(v+w_1,w_2)+\frac{\pi}{2}H(w_2,w_2)\right)\\
&\qquad \widetilde{\chi}(w_1)\exp\left(\pi H(v,w_1)+\frac{\pi}{2}H(w_1,w_1)\right)\exp\left(-\pi H(w_1,w_2)+\pi H(w_2,w_1)\right)\\
&= \chi(w_1,w_2)e_{w_1}(v+w_2)e_{w_2}(v)
\end{aligned}$$

(ii) From (i) one has $e_{w_1}(v+w_2)^{-1} = \chi(w_1,w_2)e_{w_1+w_2}(v)^{-1}e_{w_2}(v)$. Thus, by (4.6)

$$\begin{aligned}
\vartheta_D^{+w_1}(v+w_2) &= e_w(v+w_2)^{-1}\vartheta_D(v+w_1+w_2)\\
&= \chi(w_1,w_2)e_{w_2}(v)e_{w_1+w_2}(v)^{-1}\vartheta_D(v+w_1+w_2)\\
&= \chi(w_1,w_2)e_{w_1}(v)\vartheta_D^{+(w_1+w_2)}(v)
\end{aligned}$$

(iii) and (vi) follow easily from (ii).

□

4.1.12. Abelian varieties and the Lefschetz embedding

In this last part, we introduce general *Abelian varieties* and highlight one of the most classical and known applications of the theory of theta functions, known as the Lefschetz theorem.

Definition 4.1.13 (Abelian varieties)**.** Let $\mathbb{T} = {}^{V}\!/_{\Gamma}$ be a complex torus of dimension g, $\mathcal{L} \in \mathrm{Pic}(\mathbb{T})$ and $H = c_1(\mathcal{L})$ its first Chern class.

(i) If the hermitian form H with $E(\Gamma \times \Gamma) \subseteq \mathbb{Z}$ is positive definite, it is called a **polarization** of the complex torus $\mathbb{T}$.
If, moreover, H is unimodular, the polarization is said to be *principal.*

(ii) A complex torus $\mathbb{T}$ admitting a polarization H is called an **Abelian variety**.

By abuse of notation, sometimes one calls the line bundle $\mathcal{L} = \mathcal{L}(H,\chi)$ -whose hermitian form is positive definite- a polarization, and the pair $(X = \mathbb{T}, \mathcal{L})$ an Abelian variety. The next section will provide (by construction) two main examples of Abelian varieties: One important example being elliptic curves, which are 1-dimensional principally polarised Abelian varieties (or p.p.a.v. for short).

Theorem 4.1.14 (Lefschetz embedding). *Let $\mathcal{L}$ be a non-degenerate holomorphic line bundle on a g-dimensional complex torus $\mathbb{T}$. For $g \geq 3$, the global sections of $\mathcal{L}$ define an analytic embedding of $\mathbb{T}$ as a complex sub-manifold of $\mathbb{P}^m$ for some m.*

Proof. See for example [Lan82](IV, §5 Theorem 8) □

In other words, for all $g \geq 3$, a choice of a polarisation on $\mathbb{T}$ makes $\phi_{\mathcal{L}}(\mathbb{T})$ into an *algebraic* projective variety of $\mathbb{P}^m$ by Chow's theorem ([PG78](Chapter 1 § 3)). In [VdV75], one has more information about this embedding:

Theorem 4.1.15. *No Abelian variety of dimension $g \geq 3$ can be embedded into $\mathbb{P}^{2g}$*

In particular, from the proof of this theorem, one recovers a very well known result: One sees that the only possibility for Abelian varieties of dimension $g = 1$ i.e. elliptic curves over $\mathbb{C}$, are smooth algebraic cubic curves in $\mathbb{P}^2$ of the form

$$E : y^2 + (a_1 x + a_3)y = x^3 + a_2 x^2 + a_4 x + a_6$$

For $g = 2$, 2-dimensional Abelian varieties are called Abelian surfaces, and the only ones embeddable in $\mathbb{P}^4$ are Abelian surfaces of degree 10 (see for example [HM73] for details) the rest are all embedded in $\mathbb{P}^5$.

4.2. The Poincaré bundle of an Abelian variety

To any Abelian variety $(X, \mathcal{L})$, there is a uniquely determined line bundle $\mathcal{P}$ on $X \times X^\vee$, to which one can associate the **dual** Abelian variety $(X^\vee, \mathcal{P})$ as follow:

4.2.1. The Poincaré bundle

For a $\mathbb{C}$-anti-linear form $l : V \to \mathbb{C}$, define the complex *dual torus* to be

$$\mathbb{T}^\vee := {\mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})}\big/_{\Gamma^\vee}$$

where $\Gamma^\vee$ is the *dual lattice* defined as

$$\Gamma^\vee := \{ l \in \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}) \mid \langle l, \Gamma \rangle := \Im\, l(\Gamma) \subseteq \mathbb{Z} \}.$$

It is, indeed, a lattice in the $\mathbb{C}$-vector space $\overline{V}^\vee := \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$ as one has a canonical $\mathbb{R}$-pairing:

$$\begin{aligned} \langle \cdot, \cdot \rangle \, : \overline{V}^\vee \times V &\longrightarrow \mathbb{R} \\ \langle l, v \rangle &\longmapsto \Im (\, l(v) \,) \end{aligned}$$

where, the $\mathbb{R}$-vector space structure of $\overline{V}^\vee$ is given by the isomorphism

$$\begin{aligned} \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \\ l &\longmapsto \Im (\, l \,) \\ (l : v \mapsto -g(iv) + ig(v)) &\longleftarrow g \end{aligned}$$

One has a canonical homomorphism

$$\begin{aligned}\overline{V}^\vee &\longrightarrow \mathrm{Hom}(\Gamma, \mathbb{S}^1)\\ l &\longmapsto \exp\left(2\pi i \left\langle l, \cdot \right\rangle\right)\end{aligned}$$

which is surjective, since $\langle \cdot, \cdot \rangle$ is non-degenerate, and has for kernel $\Gamma^\vee$. Thus it induces an isomorphism

$$\mathbb{T}^\vee \cong \mathrm{Pic}^0(\mathbb{T})$$

Definition 4.2.2 (Poincaré bundle)**.** The **Poincaré bundle** is the[‡] holomorphic line bundle $\mathcal{P}$ on $\mathbb{T} \times \mathbb{T}^\vee$ satisfying :

(i) The restriction of $\mathcal{P}$ to $\mathbb{T} \times \{\mathcal{L}\}$ is isomorphic to $\mathcal{L}$ for every $\mathcal{L} \in \mathbb{T}^\vee$.

(ii) The restriction of $\mathcal{P}$ to $\{0\} \times \mathbb{T}^\vee$ is trivial.

Define a hermitian form $H_{\mathrm{can}} \in \mathrm{Her}(V \oplus \overline{V}^\vee)$ by

$$\begin{aligned}H_{\mathrm{can}} : V \oplus \overline{V}^\vee \times V \oplus \overline{V}^\vee &\longrightarrow \mathbb{C}\\ \big((v,l),(v',l')\big) &\longmapsto l(v') + \overline{l'(v)}\end{aligned}$$

where, clearly, $E(\Gamma \oplus \Gamma^\vee, \Gamma \oplus \Gamma^\vee) := \Im\, H_{\mathrm{can}}(\Gamma \oplus \Gamma^\vee, \Gamma \oplus \Gamma^\vee) \subseteq \mathbb{Z}$. Construct a semi-character associated to H_{can} in the following way:

$$\begin{aligned}\chi_{\mathrm{can}} : \Gamma \oplus \Gamma^\vee &\longrightarrow \mathbb{S}^1\\ (\gamma, l_0) &\longmapsto \exp\left(\pi i \Im\, l_0(\gamma)\right).\end{aligned}$$

By Appell-Humbert (theorem 4.1.4), the pair $(H_{\mathrm{can}}, \chi_{\mathrm{can}})$ defines a line bundle $\mathcal{P}$ on $\mathbb{T} \times \mathbb{T}^\vee$ with associated multiplier system $\left(e_{(\gamma,l_0)}\right)_{(\gamma,l_0)\in\Gamma\oplus\Gamma^\vee}$, given by:

$$\begin{aligned}e_{(\gamma,l_0)}(v,l) &= \chi_{\mathrm{can}}(\gamma,l_0)\exp\left(\pi H_{\mathrm{can}}\left[(v,l),(\gamma,l_0)\right] + \frac{\pi}{2}H_{\mathrm{can}}\left[(\gamma,l_0),(\gamma,l_0)\right]\right)\\ &= \exp\left(\pi i \Im\, l_0(\gamma)\right)\exp\left(\pi\left[l(\gamma) + \overline{l_0(v)}\right] + \frac{\pi}{2}\left[l_0(\gamma) + \overline{l_0(\gamma)}\right]\right)\end{aligned}$$

- Let $\mathcal{L} \in \mathrm{Pic}^0(\mathbb{T})$. From (4.3), there exists some $l \in \overline{V}^\vee$ such that

  $$\mathcal{L} = \mathcal{L}(0, \alpha_v(\gamma)) \qquad \text{where } \alpha_v(\gamma) = \exp\left(2\pi i E(v,\gamma)\right).$$

  Consider the restriction of $\mathcal{P}$ on $\mathbb{T} \times \{\mathcal{L}\}$: then, the associated multipliers $\left(e_{(\gamma,0)}\right)_{\gamma\in\Gamma}$ are given by

  $$e_{(\gamma,0)}(v,l) = \chi_{\mathrm{can}}(\gamma,0)\exp\left(\pi l(\gamma)\right) = \exp\left(\pi l(\gamma)\right).$$

[‡]This will soon be justified, see also B.2.15

Again, by remark 4.1.2, one can freely multiply both sides by a coboundary $\frac{h(v+\gamma)}{h(v)}$. Choose $h : v \longmapsto \exp\left(\pi \overline{l(v)}\right)$, this gives

$$\Leftrightarrow \qquad e_{(\gamma,0)}(v,l)\exp\left(\pi\overline{l(v+\gamma)}\right)^{-1}\exp\left(\pi\overline{l(v)}\right) = \exp\pi\left(l(\gamma) - \overline{l(v+\gamma)} + \overline{l(v)}\right)$$
$$= \exp\left(2\pi i \Im\, l(\gamma)\right) = \alpha_v(\gamma)$$

which is the multiplier for the line bundle $\mathcal{L}$. Hence, this induces an isomorphism

$$\mathcal{P}_{|\mathbb{T}\times\{\mathcal{L}\}} \cong \mathcal{L}$$

- Similarly, for $l \in \overline{V}^{\vee}$, consider the restriction of $\mathcal{P}$ on $\{0\}\times\mathbb{T}^{\vee}$: the associated multipliers $\left(e_{(0,l_0)}\right)_{l_0\in\mathbb{T}^{\vee}}$ are given by

$$e_{(0,l_0)}(0,l) = 1$$

which is, nothing but the multiplier for the trivial line bundle on $\mathcal{P}_{|\{0\}\times\mathbb{T}^{\vee}}$.

Hence, a Poincaré bundle always exists on $\mathbb{T}\times\mathbb{T}^{\vee}$, and is non-degenerate by construction. Moreover, the *Seesaw principle*[§] shows that it is uniquely determined up to isomorphisms. Thus every line bundle $\mathcal{L}\in \mathrm{Pic}^0(\mathbb{T})$ is isomorphic to the restriction $\mathcal{P}_{|\mathbb{T}\times\{\mathcal{L}\}}$ for a **unique** $\mathcal{L}\in\mathbb{T}^{\vee}$.

Example 4.2.3 (The Mumford bundle). Let $(X,\mathcal{L})$ be an Abelian variety. Consider the morphisms:

$$\begin{array}{rcl} m : X\times X & \longrightarrow & X \\ (x,y) & \longmapsto & x+y \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p_i : X\times X & \longrightarrow & X \\ (x,y) & \longmapsto & \begin{cases} x & \text{if } i=1, \\ 0 & \text{if } i=2. \end{cases} \end{array}$$

and define the *Mumford line bundle* to be

$$\mathcal{M} := [m]^*\mathcal{L}\otimes p_1^*\mathcal{L}^{-1}\otimes p_2^*\mathcal{L}^{-1} \cong (\mathrm{id}\times\phi_{\mathcal{L}})^*\mathcal{P} \qquad \text{(as line bundles on } X\times X) \tag{4.7}$$

First, observe that, for $y=\phi_{\mathcal{L}}(x)$ where $x\in X$:

$$\mathcal{M}_{|X\times\{x\}} = t_x^*\mathcal{L}\otimes\mathcal{L}^{-1}\in \mathrm{Pic}^0(X)$$
$$\left(= (\mathrm{id}\times\phi_{\mathcal{L}})^*\mathcal{P}_{|X\times\{x\}} = \mathcal{P}_{|X\times\{y\}}\right)$$

since, by theorem 4.1.6 and for $x'\in X$:

$$t_{x'}^*(t_x^*\mathcal{L}\otimes\mathcal{L}^{-1}) = t_{x'+x}^*\mathcal{L}\otimes t_{x'}^*(\mathcal{L}^{-1}) \cong t_{x'}^*\mathcal{L}\otimes t_x^*\mathcal{L}\otimes\mathcal{L}^{-1}\otimes t_{x'}^*\mathcal{L}\cong t_x^*\mathcal{L}\otimes\mathcal{L}^{-1}$$
$$\Rightarrow \quad \mathrm{Im}\,\phi_{\mathcal{L}}\in\mathrm{Pic}^0(X)$$

[§]See lemma B.2.7 in appendix B.

And similarly,

$$\mathcal{M}_{|\{0\}\times X^\vee} = \mathcal{L}\otimes\mathcal{L}^{-1} \cong \mathcal{O}_X \cong \mathcal{P}_{|\{0\}\times X^\vee}$$

Explicitly, one has $\mathcal{M} = \mathcal{M}(H_{\mathcal{M}}, \chi_{\mathcal{M}})$ where for $z_i, z_i' \in V$ and $\gamma, \gamma' \in \Gamma$:

$$H_{\mathcal{M}}\left((z_1,z_2),(z_1',z_2')\right) = H_{\mathcal{L}}\left(z_1+z_1', z_2+z_2'\right) - H_{\mathcal{L}}(z_1,z_1') - H_{\mathcal{L}}(z_1,z_1')$$
$$\chi_{\mathcal{M}}(\gamma+\gamma') = \chi_{\mathcal{L}}(\gamma+\gamma')\chi_{\mathcal{L}}(\gamma)^{-1}\chi_{\mathcal{L}}(\gamma')^{-1} = \alpha_v\left(\frac{\gamma}{2}\right)$$

Remark 4.2.4 (Universal property of dual Abelian varieties). The Poincaré bundles are universal objects in the following sense: For an Abelian variety $(X,\mathcal{M})$ and a scheme T with the line bundle $\mathcal{M}'$ on $X\times T$ satisfying (i) and (ii) in definition 4.2.2, there exists a unique morphism $f: T \longrightarrow X^\vee$ such that

$$(\mathrm{id}\times f)^*\mathcal{P} = \mathcal{M}'$$

See B.2.15 in appendix B.

4.2.5. The case of 1-dimensional complex Abelian varieties: elliptic curves

Let $X(\mathbb{C}) = E(\mathbb{C})$ be an elliptic curve, with origin $O = \{0\}$ and let $\mathcal{L} = \mathcal{L}([0])$ be the line bundle associated to the divisor $[0] \in \mathrm{Div}(E)$.

Recall that the Weierstrass σ-function:

$$\sigma(z,\Gamma) = z\prod_{\gamma\in\Gamma\smallsetminus\{0\}}\left(1-\frac{z}{\gamma}\right)\exp\left(\frac{z}{\gamma}+\frac{z^2}{2\gamma^2}\right)$$

satisfies the logarithmic derivative of the Weierstrass's ζ-function in eq. (2.17) (for $z \in \mathbb{C}\smallsetminus\Gamma$)

$$\zeta(z,\Gamma) = \frac{\sigma'(z)}{\sigma(z)}$$

Hence, one has that

$$\frac{\frac{d}{dz}\left(\frac{\sigma(z+\gamma)}{\sigma(z)}\right)}{\frac{\sigma(z+\gamma)}{\sigma(z)}} = \frac{d}{dz}\left(\ln\frac{\sigma(z+\gamma)}{\sigma(z)}\right) = \eta(\gamma)$$
$$\Rightarrow \quad \frac{\sigma(z+\gamma)}{\sigma(z)}e^{-\eta(\gamma)z} = e^{\delta}$$

where $\eta:\Gamma\to\mathbb{C}$ is the quasi-period $\eta(\gamma) := \zeta(z+\gamma)-\zeta(z)$ and is independent[¶] from z, and some constant δ. Define

$$\chi_{[0]}(\gamma) := \frac{\sigma(z+\gamma)}{\sigma(z)}e^{-\eta(\gamma)(z+\frac{\gamma}{2})} \qquad \text{(which does clearly not depend on } z\text{)}$$

[¶]Indeed, one easily checks that for $\gamma\in\Gamma$: $\frac{d}{dz}\left(\zeta(z+\gamma)-\zeta(z)\right) = -\wp(z+\gamma)+\wp(z) = 0$

Let $\gamma \in \Gamma$. If $\gamma \notin 2\Gamma$, then for $z = -\frac{\gamma}{2}$:

$$\chi_{[0]}(\gamma) = \frac{\sigma(\frac{\gamma}{2})}{\sigma(-\frac{\gamma}{2})} = -1 \quad \text{as } \sigma \text{ is odd.}$$

If $0 \neq \gamma \in 2\Gamma$, then there exists an order n such that $\gamma \notin 2^n\Gamma$ and $\gamma \in 2^k\Gamma \quad \forall 0 \geq k < n$. Thus, for $\gamma' = \frac{\gamma}{2^n}$:

$$\chi_{[0]}(\gamma) = \chi_{[0]}(2^n\gamma') \underset{\substack{\uparrow \\ \text{since } \eta \text{ is additive}}}{=} \chi_{[0]}(\gamma')^{2^n} = 1 \quad \text{as } \gamma' \notin \Gamma.$$

and one has

$$\sigma(z+\gamma) := \chi_{[0]}(\gamma)e^{-\eta(\gamma)(z+\frac{\gamma}{2})}\sigma(z)$$

where $\chi_{[0]}$ is a semi-character

$$\begin{aligned} \chi_{[0]}(\gamma) \,:\, \Gamma &\longrightarrow \{\pm 1\} \subset \mathbb{S}^1 \\ \gamma &\longmapsto \begin{cases} 1 & \text{if } \gamma \in 2\Gamma, \\ -1 & \text{if } \gamma \notin 2\Gamma. \end{cases} \end{aligned}$$

Now, observe that the quasi-period η is clearly $\mathbb{Z}$-linear and thus, entirely determined by the periods (ω_1, ω_2). Moreover, by eq. (2.17), one sees that for $\gamma = m_1\omega_1 + m_2\omega_2 \in \Gamma$:

$$\begin{aligned} \eta(\gamma) &= \zeta(z+\gamma) - \zeta(z) = E_1(z + m_1\omega_1 + m_2\omega_2) - E_1(z, \omega_1, \omega_2) + e_2\gamma \\ &= -\frac{2\pi i}{\omega_1}m_1 + e_2\gamma \qquad\qquad \text{by (2.19)} \\ &= \frac{\overline{\omega_1}}{A\omega_1}\gamma - \frac{2\pi i}{\omega_1}m_1 + e_2^*\gamma \qquad \text{by (2.27)} \\ &= \frac{\overline{\omega_1}}{A} + e_2^*\gamma \end{aligned}$$

Hence, define

$$\theta(z) := \exp\left(-\frac{e_2^*}{2}z^2\right)\sigma(z).$$

One easily sees that

$$\begin{aligned} \theta(z+\gamma) &= \chi_{[0]}(\gamma)\exp\left(\frac{z\overline{\gamma}}{A} + \frac{|\gamma|^2}{2A}\right)\theta(z) \\ &= \chi_{[0]}(\gamma)\exp\left(\pi H_{[0]}(z,\gamma) + \frac{1}{2}H_{[0]}(\gamma,\gamma)\right)\theta(z) \end{aligned}$$

where

$$H_{[0]}(z_1, z_2) = \frac{z_1\overline{z_2}}{\pi A} \in \mathrm{Her}(V) \quad \text{and} \quad \chi_{[0]}(\gamma) = \pm 1.$$

Thus, the holomorphic function θ is a reduced theta function associated to the divisor $[0]$ i.e. the line bundle $\mathcal{L}(H_{[0]}, \chi_{[0]})$.

Remark 4.2.6. Recall that for $x \in E(\mathbb{C})$

$$\begin{aligned} t_x^{-1}(p=0) = x \quad &\Rightarrow \quad t_x^* \mathscr{O}_E(p) \otimes \mathscr{O}(-p) \cong \mathscr{O}_E(x-p) \\ (\text{as } \ker(\phi_{\mathcal{L}\sim[p=0]}) \text{ is finite}) \quad &\Rightarrow \quad \mathrm{Pic}^0(E) = \{[\mathscr{O}_E(x-p)] \mid x \in E(\mathbb{C})\} \end{aligned}$$

More generally, for a divisor $D = \sum_i n_i p_i \in \mathrm{Div}(E)$ one has

$$\mathrm{Pic}^0(E) = \{[\mathscr{O}_E(D)] \mid \deg(D) = 0\}$$

Now, let $D = [0]$. Taking the canonical polarization $\mathcal{L} = \mathcal{L}([0])$, one has the identification $E^\vee \cong E$ (since the polarization is principal) and

$$\mathcal{M} \cong \mathscr{O}_{E\times E}\left(m^{-1}(O) - p_1^{-1}(O) - p_2^{-1}(O)\right). \tag{4.8}$$

Hence, the Poincaré bundle on $E \times E$ is given by

$$\mathcal{P}_E \cong \mathscr{O}_{E\times E}\left(\Delta - p_1^*(O) - p_2^*(O)\right) \qquad \text{where } \Delta = \ker(m : E \times E \longrightarrow E).$$

Explicitly, one recovers the results found in the previous subsection as $\mathcal{P}_E = \mathcal{P}_E(H_{\mathcal{P}}, \chi_{\mathcal{P}})$ and

$$\begin{aligned} H_{\mathcal{P}} &= m^* H_{[0]} - p_1^* H_{[0]} - p_2^* H_{[0]} \quad \text{and} \quad & \chi_{\mathcal{P}} &= m^* \chi_{[0]} \cdot p_1^* \chi_{[0]}^{-1} \cdot p_2^* \chi_{[0]}^{-1} \\ (&= H_{\mathrm{can}}) & (&= \chi_{\mathrm{can}}) \end{aligned}$$

where

$$\begin{aligned} H_{\mathcal{P}}\left((z_1,z_2),(z_1',z_2')\right) &= H_{[0]}\left(z_1+z_1', z_2+z_2'\right) - \frac{z_1\overline{z_1'}}{\pi A} - \frac{z_2\overline{z_2'}}{\pi A} = \frac{z_1\overline{z_2'} + z_1'\overline{z_2}}{\pi A} \\ \chi_{\mathcal{P}}(\gamma,\gamma') &= \chi_{[0]}(\gamma+\gamma')\chi_{[0]}(\gamma)^{-1}\chi_{[0]}(\gamma')^{-1} = \exp\left(\pi i \frac{1}{\pi A}\Im\left(\gamma\overline{\gamma'}\right)\right) \\ &= \exp\left(\frac{\gamma\overline{\gamma'} - \overline{\gamma}\gamma'}{2A}\right) \end{aligned}$$

Thus, $\mathcal{P}_E$ has the associated reduced theta function ϑ, satisfying the transformation formula:

$$\vartheta(z+\gamma, z'+\gamma') = e_{(\gamma,\gamma')}\vartheta(z,z')$$

where, the associated multiplier system $\left(e_{(\gamma,\gamma')}\right)_{(\gamma,\gamma')\in\Gamma\oplus\Gamma}$ is given by:

$$\begin{aligned} e_{(\gamma,\gamma')}(z,z') &= \chi_{\mathcal{P}}(\gamma,\gamma')\exp\left(\pi H_{\mathcal{P}}\left[(z,z'),(\gamma,\gamma')\right] + \frac{\pi}{2}H_{\mathcal{P}}\left[(\gamma,\gamma'),(\gamma,\gamma')\right]\right) \\ &= \exp\left(\frac{\gamma\overline{\gamma'}}{A}\right)\exp\left(\frac{z\overline{\gamma'} + z'\overline{\gamma}}{A}\right) \end{aligned}$$

4.2.7. The Kronecker theta function Θ

So far, given an elliptic curve $E(\mathbb{C}) = {}^{\mathbb{C}}/_{\Gamma}$ with Poincaré bundle $\mathcal{P}_E$, one has an associated reduced theta function given by a meromorphic section $s : E(\mathbb{C}) \times E(\mathbb{C}) \longrightarrow \mathcal{P}_E$, i.e. a meromorphic function $\vartheta : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ satisfying the transformation formula:

$$\vartheta(z + \gamma, z' + \gamma') = \exp\left(\frac{\gamma\overline{\gamma'}}{A}\right)\exp\left(\frac{z\overline{\gamma'} + z'\overline{\gamma}}{A}\right)\vartheta(z, z') \tag{4.9}$$

Remark 4.2.8. Let $h : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying the transformation formula (4.9), and let $z_0, z_0' \in \mathbb{C}$ and $\gamma, \gamma' \in \Gamma$ such that $\gamma' = -\gamma$. One easily sees that

$$\begin{aligned}\left|h(z_0 + \gamma, z_0' + \gamma')\right| &= \left|\exp\left(\frac{-\gamma\overline{\gamma} - z_0\overline{\gamma} + z_0'\overline{\gamma}}{A}\right)h(z_0, z_0')\right| \\ &= \exp\left(-\frac{|\gamma|^2 + \Re(z_0 - z_0')\overline{\gamma}}{A}\right)\left|h(z_0, z_0')\right| \xrightarrow[|\gamma|\to\infty]{} 0\end{aligned}$$

This implies that the holomorphic function $f : t \longmapsto h(z_0 + t, z_0' - t)$ is bounded, and, by Liouville's theorem, that

$$h(z_0, z_0') = 0.$$

Hence, any holomorphic function $h(z, z')$ on $\mathbb{C} \times \mathbb{C}$ satisfying (4.9) for any $\gamma, \gamma' \in \Gamma$ is identically null.

This shows that $\mathcal{P}_E$ has no non-zero holomorphic section. However, a divisor $D \in \operatorname{Div}(E)$ always define a meromorphic section s_D on $\mathcal{P}_E$ which (in the light of (4.7) and (4.8)) is given by

$$s_D = m^* s_{[0]} \otimes p_1^* s_{[0]} \otimes p_2^* s_{[0]} \qquad \text{for a section } s_{[0]} \text{ of } \mathcal{L}([0])$$

Definition 4.2.9 (Kronecker theta function)**.** Let $E(\mathbb{C}) = {}^{\mathbb{C}}/_{\Gamma}$ be an elliptic curve, $D \in \operatorname{Div}(E)$. We define the **Kronecker theta function** to be the reduced theta function corresponding to the divisor D, i.e. the meromorphic function

$$\Theta(z, z') := \frac{\theta(z + z')}{\theta(z)\theta(z')} \qquad \text{where} \qquad \theta(z) = \exp\left(-\frac{e_2^*}{2}z^2\right)\sigma(z) \tag{4.10}$$

At first, one clearly sees that

$$\Theta(z, z') = \exp\left(-e_2^* z z'\right)\frac{\sigma(z + z')}{\sigma(z)\sigma(z')} \qquad \text{has residues } \operatorname{res}_{z=0}(\Theta, z) = \operatorname{res}_{z'=0}(\Theta, z') = 1$$

Recall from (2.38) in chapter 2, that for a positive integer a and complex numbers $z, z' \in \mathbb{C}$:

$$\begin{aligned}\theta_{a,t}(z, z') :&= \sum_{\gamma\in\Gamma}\exp\left(-t|z + \gamma|^2\right)\langle\gamma, z'\rangle(\overline{z} + \overline{\gamma})^a \\ &= \sum_{\gamma\in\Gamma}\exp\left(-t|z + \gamma|^2\right)\exp\left(\frac{\gamma\overline{z'} - z'\overline{\gamma}}{A}\right)(\overline{z} + \overline{\gamma})^a\end{aligned}$$

where $t \in \mathbb{R}_{>0}$. Pose

$$\vartheta_{a,t}(z,z') := \exp\left(\frac{z\overline{z'}}{A}\right)\theta_{a,t}(z,z')$$

One easily sees that

$$\begin{aligned}\theta_{a,t}(z+\gamma,z'+\gamma') &= \theta_{a,t}(z+\gamma,z')\exp\left(\frac{\gamma\overline{\gamma'}-\gamma'\overline{\gamma}}{A}\right)\\ &= \theta_{a,t}(z+\gamma,z') && \text{by (iii) in §2.3.1}\end{aligned}$$

and

$$\begin{aligned}\exp\left(\frac{(z+\gamma)(\overline{z'+\gamma'})}{A}\right) &= \exp\left(\frac{(z+\gamma)\overline{z'}-z'(\overline{z+\gamma})}{A}\right)\exp\left(\frac{(z+\gamma)\overline{\gamma'}+z'(\overline{z+\gamma})}{A}\right)\\ &= \chi(z+\gamma,z')\exp\left(\frac{z\overline{\gamma'}+z'\overline{\gamma}}{A}\right)\exp\left(\frac{\gamma\overline{\gamma'}+z'\overline{z}}{A}\right)\end{aligned}$$

Thus, it easily follows that

$$\begin{aligned}\vartheta_{a,t}(z+\gamma,z'+\gamma') &= \exp\left(\frac{(z+\gamma)(\overline{z'+\gamma'})}{A}\right)\theta_{a,t}(z+\gamma,z'+\gamma')\\ &= \exp\left(\frac{\gamma\overline{\gamma'}}{A}\right)\exp\left(\frac{z\overline{\gamma'}+z'\overline{\gamma}}{A}\right)\theta_{a,t}(z+\gamma,z')\chi(z+\gamma,z')\exp\left(\frac{z'\overline{z}}{A}\right).\end{aligned}$$

Since $\widehat{\theta}_{a,t,z'}(z) := \theta_{a,t}(z,z')\chi(z,z')$ is Γ-periodic, one gets that

$$\begin{aligned}\vartheta_{a,t}(z+\gamma,z'+\gamma') &= e_{(\gamma,\gamma')}(z,z')\theta_{a,t}(z,z')\chi(z,z')\exp\left(\frac{z'\overline{z}}{A}\right)\\ &= e_{(\gamma,\gamma')}(z,z')\exp\left(\frac{z\overline{z'}}{A}\right)\theta_{a,t}(z,z')\\ &= e_{(\gamma,\gamma')}(z,z')\vartheta_{a,t}(z,z') \qquad (4.11)\end{aligned}$$

Hence, the real analytic function $\vartheta_{a,t}$ defines a smooth section of $\mathcal{P}_E$. Recall from eq. (2.37) that for $z \in \mathbb{C} \smallsetminus \Gamma$ and $z' \in \mathbb{C}$, $K_a(z,z',s)$ is $\mathscr{C}^\infty$ and satisfies:

$$K_a(z,z',s) = \frac{1}{\Gamma(s)}\int_0^\infty \sum_{\gamma\in\Gamma}\theta_{a,t}(z,z')t^{s-1}\,dt$$

Define

$$\kappa_a(z,z',s) := \exp\left(\frac{z\overline{z'}}{A}\right)K_a(z,z',s) = \frac{1}{\Gamma(s)}\int_0^\infty \sum_{\gamma\in\Gamma}\vartheta_{a,t}(z,z')t^{s-1}\,dt \qquad (4.12)$$

For $a = s = 1$ and by (2.43), one sees that

$$\frac{\partial}{\partial\overline{z}}\kappa_1(z,z',1) = 0$$

Hence, $\kappa_1(z,z',1)$ is holomorphic in $z \in \mathbb{C} \setminus \Gamma$. Note that

$$\begin{aligned}
\kappa_1(z,z',1) &= \exp\left(\frac{z\overline{z'}}{A}\right) K_1(z,z',1) \\
&= \exp\left(\frac{z\overline{z'}}{A}\right) K_1(z',z,1) \exp\left(\frac{z'\overline{z} - z\overline{z'}}{A}\right) \qquad \text{(by the functional equation (2.36))} \\
&= \kappa_1(z',z,1) \\
\Leftrightarrow \qquad & \frac{\partial}{\partial\overline{z'}}\kappa_1(z,z',1) = \frac{\partial}{\partial\overline{z'}}\kappa_1(z',z,1) = 0
\end{aligned}$$

Thus, $\kappa_1(z,z',1)$ is also holomorphic in $z' \in \mathbb{C} \setminus \Gamma$. It has possible poles at $z \in \Gamma$ or $z' \in \Gamma$ since by (2.41)

$$\begin{aligned}
K_1(z,z',1) &= I_{A^{-1}}(1,z,z',1) + \langle z', z \rangle I_{A^{-1}}(1,z',z,1) \\
&= \int_{A^{-1}}^{\infty} \theta_{1,t}(z,z')\,dt + \langle z', z \rangle \int_{A^{-1}}^{\infty} \theta_{1,t}(z',z)\,dt
\end{aligned}$$

If z or z' is in Γ, then by (4.11) one can assume that it is equal to 0. One then see that

$$\begin{aligned}
\kappa_1(z,z',1) - \int_{A^{-1}}^{\infty} \exp\left(-t|z|^2 + \frac{z\overline{z'}}{A}\right)\overline{z}\,dt - \langle z', z \rangle \int_{A^{-1}}^{\infty} \exp\left(-t|z'|^2 + \frac{z\overline{z'}}{A}\right)\overline{z'}\,dt \\
= \int_{0}^{\infty} \sum_{\substack{\gamma\in\Gamma \\ \gamma\neq 0}} \exp\left(-t|z+\gamma|^2\right) \exp\left(\frac{\gamma\overline{z} - z\overline{\gamma} + z\overline{z'}}{A}\right)(\overline{z} + \overline{\gamma})\,dt
\end{aligned}$$

is real analytic, and thus $zz'\kappa_1(z,z',1)$ is clearly holomorphic (and nonzero) in a neighbourhood of 0. Moreover, one also has that

$$\begin{cases}
-z \int_{A^{-1}}^{\infty} \exp\left(-t|z|^2 + \frac{z\overline{z'}}{A}\right)\overline{z}\,dt & = \exp\left(-\frac{|z|^2}{A} + \frac{z\overline{z'}}{A}\right) \xrightarrow[z\to 0]{} 1 \\
-z' \int_{A^{-1}}^{\infty} \exp\left(-t|z'|^2 + \frac{z\overline{z'}}{A}\right)\overline{z'}\,dt & = \exp\left(-\frac{|z'|^2}{A} + \frac{z\overline{z'}}{A}\right) \xrightarrow[z'\to 0]{} 1
\end{cases}$$

and thus,

$$\mathrm{res}_{z=0}(\kappa_1, z) = \mathrm{res}_{z'=0}(\kappa_1, z') = 1$$

Summing up, $\kappa_1(z,z',1)$ defines a meromorphic section of the Poincaré bundle $\mathcal{P}_E$ where $E(\mathbb{C})$ is an elliptic curve. It has simple poles at $z = 0$ and $z' = 0$ with residue equal to 1. These correspond to the divisors $O \times E$ and $E \times O$, respectively. Thus, $\kappa_1(z,z',1)$ and $\Theta(z,z')$ BOTH define a meromorphic section of $\mathcal{P}_E$ with the same simple pole and identical residues on it. This means that the function $\kappa_1(z,z',1) - \Theta(z,z')$ defines a holomorphic section of $\mathcal{P}_E$ which, by remark 4.2.8, is identically null. This finally relates the Kronecker theta function Θ to the the Eisenstein-Kronecker-Lerch series:

Theorem 4.2.10 (Kronecker). *The Kronecker theta function associated to the divisor $D = \Delta - (O \times E) - (E \times O)$ is given by*

$$\Theta(z,z') = \frac{\theta(z+z')}{\theta(z)\theta(z')} = \exp\left(\frac{z\overline{z'}}{A}\right) K_1(z,z',1)$$

This will be the key result in order to prove the main theorem of this thesis in the next section. We end this section by the following remark: recall from §4.1.10 that given a reduced theta function ϑ_D corresponding to some meromorphic sections s, one defines its translation to be

$$\vartheta_D^{+w}(v) := e_w(v)^{-1}\vartheta_D(v+w)$$

We would like to reproduce this method in order to define a translation for the Kronecker theta function. Consider once again the Mumford bundle

$$\mathcal{M} = [m]^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} = \mathcal{L}(H_{\mathcal{M}}, \chi_{\mathcal{M}})$$

where clearly

$$\chi_{\mathcal{M}}(v,w) = \exp\left(2\pi i E\left(v, \frac{w}{2}\right)\right).$$

Extend χ into a map

$$\begin{aligned} \widetilde{\chi}_{\mathcal{M}} : V \times V &\longrightarrow \mathbb{C}^\times \\ (v,w) &\longmapsto \exp\left(2\pi i E\left(v, \frac{w}{2}\right)\right) \end{aligned}$$

and define the map $e : V^2 \times V^2 \longrightarrow \mathbb{C}^\times$ by

$$e_{(w,w')}(v,v') := \widetilde{\chi}(w,w')\exp\left(\pi H_{\mathcal{M}}\left((v,w),(v',w')\right) + \frac{\pi}{2}H_{\mathcal{M}}\left((w,w'),(w,w')\right)\right).$$

This defines the translation of the Kronecker theta function

$$\Theta_{(w,w')}(z,z') := \Theta(z,z')^{+(w,w')} = e_{(w,w')}(v,v')^{-1}\Theta(z+w,z'+w'). \tag{4.13}$$

In the case of an elliptic curve $E(\mathbb{C})$, one has:

$$\Theta_{(w,w')}(z,z') = \exp\left(-\frac{w\overline{w'}}{A}\right)\exp\left(-\frac{z\overline{w'}+z'\overline{w}}{A}\right)\Theta(z+w,z'+w'). \tag{4.14}$$

4.3. The generating function of the Eisenstein-Kronecker numbers

Let $a \geq 1, b > 0$ be positive integer and z_0, z_0' be fixed complex numbers. Recall the series expansion for the Eisensitein-Kronecker-Lerch series given by (2.35) :

$$\Gamma(s)K_a^*(z_0,z_0',s) = I_{A^{-1}}(a,z_0,z_0',s) + A^{a+1-2s}I_{A^{-1}}(a+1,z_0',z_0,1-s)\langle\, z_0', z_0 \,\rangle_\Gamma$$

where

$$I_{A^{-1}}(a, z_0, z_0', s) = \int_{A^{-1}}^{\infty} \theta_{a,t}(z_0, z_0') t^{s-1} \, dt$$

The aim of this section, and the main result of this thesis, is that the translation of the Kronecker theta function $\Theta_{z_0,z_0'}(z, z')$ is a generating function for the Eisenstein-Kronecker numbers $e^*_{a,b}(z_0, z_0')$:

Theorem 4.3.1 ([BK⁺10])**.** *For $z_0, z_0' \in \mathbb{C}$, the Laurent expansion of $\Theta_{(z_0,z_0')}(z, z')$ at $(0,0)$ is given by*

$$\Theta_{(z_0,z_0')}(z, z') = \langle z_0', z_0 \rangle \frac{\delta(z_0, \Gamma)}{z} + \frac{\delta(z_0', \Gamma)}{z'} + \sum_{\substack{a,b \in \mathbb{N} \\ b \neq 0}} (-1)^{a+b-1} \frac{e^*_{a,b}(z_0, z_0')}{a! A^a} z^{b-1} z'^a$$

In particular, $\Theta_{(z_0,z_0')}(z, z')$ is the generating function of the Eisenstein-Kronecker numbers $e^*_{a,b}(z_0, z_0')$.

The proof follows mainly from theorem 4.2.10 and lemma 2.3.7. The main technicality would be to work around singularities when $z, z' \in \Gamma$. Thus, we will prove this result in two parts:

Proof. (i) Suppose that $z, z' \in \mathbb{C} \setminus \Gamma$: Recall from (4.12) that

$$\kappa_{a+b}(z, z', b) := \exp\left(\frac{z\overline{z'}}{A}\right) K_{a+b}(z, z', b).$$

By lemma 2.3.7, one sees that

$$\frac{\partial}{\partial z} \kappa_{a+b}(z, z', b) = -b\kappa_{a+b+1}(z, z', b+1)$$
$$\frac{\partial}{\partial z'} \kappa_{a+b}(z, z', b) = -A^{-1} \kappa_{a+b+1}(z, z', b)$$

Thus, by definition 2.3.8, the Taylor series of the holomorphic function $\kappa_1(z+z_0, z'+z_0', 1)$ around $(0,0)$ is given by

$$\sum_{\substack{a \geq 0 \\ b > 0}} (-1)^{a+b-1} \frac{e^*_{a+b-1}(z_0, z_0')}{a! A^a} z^{b-1} z'^a.$$

On the other hand, one has that

$$\begin{aligned}
\Theta_{(z_0,z_0')}(z, z') &= \exp\left(-\frac{z_0\overline{z_0'}}{A}\right) \exp\left(-\frac{z\overline{z_0'} + z'\overline{z_0}}{A}\right) \Theta(z + z_0, z' + z_0') \\
&= \exp\left(\frac{(z + z_0)\overline{z'} + (z' + z_0')\overline{z}}{A}\right) \exp\left(\frac{z_0'\overline{z_0}}{A}\right) \kappa_1(z + z_0, z' + z_0', 1) \\
&= u_{z,z'}(\overline{z}, \overline{z'}) \exp\left(\frac{z_0'\overline{z_0}}{A}\right) \kappa_1(z + z_0, z' + z_0', 1)
\end{aligned}$$

where $u_{z,z'}(\overline{z},\overline{z'})$ is a real-analytic function in $(z,z',\overline{z},\overline{z'})$, with $u_{z,z'}(0,0)=1$. Thus derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial z'}$ commute with the evaluation $\overline{z}=\overline{z'}=0$ and one gets

$$\begin{aligned}
&\sum_{\substack{a\geq 0\\ b>0}} \frac{1}{a!(b-1)!}\left(\frac{\partial}{\partial z}\right)^{b-1}\left(\frac{\partial}{\partial z'}\right)^{a}\left(u_{z,z'}(\overline{z},\overline{z'})\exp\left(\frac{z_0'\overline{z_0}}{A}\right)\kappa_1(z+z_0,z'+z_0',1)\right)_{\substack{z=0,z'=0\\ \overline{z}=0,\overline{z'}=0}} z^{b-1}z'^{a}\\
&\quad=\sum_{\substack{a\geq 0\\ b>0}} \frac{1}{a!(b-1)!}\exp\left(\frac{z_0'\overline{z_0}}{A}\right)\left(\frac{\partial}{\partial z}\right)^{b-1}\left(\frac{\partial}{\partial z'}\right)^{a}\left(\kappa_1(z+z_0,z'+z_0',1)\right)_{|z=0,z'=0} z^{b-1}z'^{a}\\
&\quad=\sum_{\substack{a\geq 0\\ b>0}} \frac{1}{a!(b-1)!}(-1)^{a+b-1}\frac{(b-1)!}{A^a}e^*_{a,b}(z_0,z_0')z^{b-1}z'^{a}\\
&\quad=\sum_{\substack{a\geq 0\\ b>0}} (-1)^{a+b-1}\frac{e^*_{a,b}(z_0,z_0')}{a!A^a}z^{b-1}z'^{a}.
\end{aligned}$$

(ii) Suppose $z,z'\in\Gamma$: Here, we mainly have problems in $-z_0,-z_0'$, respectively, when $z_0,z_0'\in\Gamma$. One way to counter these singularities is to introduce new auxiliary functions in order to end up with the case (i). We proceed carefully as follow: Let Γ_1,Γ_2 be two subsets of Γ, and define for $i=1,2$

$$\theta_{a,t}(z_0,z_0';\Gamma_i):=\sum_{\gamma\in\Gamma_i}\exp\left(-t|z_0+\gamma|^2\right)(\overline{z_0}+\overline{\gamma})^a\langle\gamma,z_0'\rangle_\Gamma$$

$$\widetilde{I}_{A^{-1}}(a,z_0,z_0',b-1;\Gamma_i):=\exp\left(-\frac{\overline{z_0}z_0'}{A(\Gamma)}\right)\int_{A^{-1}(\Gamma)}^{\infty}\theta_{a,t}(z_0,z_0';\Gamma_i)t^{b-2}\,dt.$$

By proposition 2.3.3, the function $\widetilde{I}_{A^{-1}}(a,z_0,z_0',b-1;\Gamma_i)$ is holomorphic in (z_0,z_0') whenever $-z_0'\notin\Gamma_i$ and one has

$$\begin{aligned}
\frac{\partial}{\partial z}\widetilde{I}_{A^{-1}}(a,z_0,z_0',b-1;\Gamma_i)&=\exp\left(-\frac{\overline{z_0}z_0'}{A(\Gamma)}\right)\int_{A^{-1}(\Gamma)}^{\infty}\frac{\partial}{\partial z}\theta_{a,t}(z_0,z_0';\Gamma_i)t^{b-2}\,dt\\
&=-\exp\left(-\frac{\overline{z_0}z_0'}{A(\Gamma)}\right)\int_{A^{-1}(\Gamma)}^{\infty}\theta_{a+1,t}(z_0,z_0';\Gamma_i)t^{b-1}\,dt\\
&=-\widetilde{I}_{A^{-1}}(a+1,z_0,z_0',b;\Gamma_i)
\end{aligned}\tag{4.15}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial z'}\widetilde{I}_{A^{-1}}(a,z_0,z_0',b-1;\Gamma_i)&=\exp\left(-\frac{\overline{z_0}z_0'}{A(\Gamma)}\right)\int_{A^{-1}(\Gamma)}^{\infty}\frac{\partial}{\partial z'}\theta_{a,t}(z_0,z_0';\Gamma_i)t^{b-2}\,dt\\
&=-A^{-1}(\Gamma)\exp\left(-\frac{\overline{z_0}z_0'}{A(\Gamma)}\right)\int_{A^{-1}(\Gamma)}^{\infty}\theta_{a+1,t}(z_0,z_0';\Gamma_i)t^{b-2}\,dt\\
&=-A^{-1}(\Gamma)\widetilde{I}_{A^{-1}}(a+1,z_0,z_0',b-1;\Gamma_i).
\end{aligned}\tag{4.16}$$

Analogously to (2.35), one defines an auxiliary function $\widetilde{K}_{a,b}$ by

$$\widetilde{K}_{a,b}(z_0, z_0'; \Gamma_1, \Gamma_2) := \frac{1}{(b-1)!}\left(\widetilde{I}_{A^{-1}}(a+b, z_0, z_0', b; \Gamma_1) + A^{a-b+1}(\Gamma)\widetilde{I}_{A^{-1}}(a+b, z_0', z_0, a+1; \Gamma_2)\right).$$

It is analytic in (z_0, z_0') whenever $(-z_0, -z_0') \notin \Gamma_1 \times \Gamma_2$ and one easily sees from (4.15) and (4.16) that

$$\begin{aligned}\frac{\partial}{\partial z}\widetilde{K}_{a,b}(z_0, z_0'; \Gamma_1, \Gamma_2) &= -b\widetilde{K}_{a,b+1}(z_0, z_0'; \Gamma_1, \Gamma_2)\\ \frac{\partial}{\partial z'}\widetilde{K}_{a,b}(z_0, z_0'; \Gamma_1, \Gamma_2) &= -A^{-1}(\Gamma)\widetilde{K}_{a+1,b}(z_0, z_0'; \Gamma_1, \Gamma_2)\end{aligned}$$

Now suppose that $z_0, z_0' \in \Gamma$, and let $\Gamma_1 := \Gamma \setminus \{-z_0\}$ and $\Gamma_2 = \Gamma \setminus \{-z_0'\}$. By (2.40), one has

$$\begin{cases}\theta_{a,t}(z_0, z_0'; \Gamma_1) &= \theta_{a,t}(z_0, z_0', \Gamma)\\ \theta_{a,t}(z_0', z_0; \Gamma_2) &= \theta_{a,t}(z_0', z_0, \Gamma)\end{cases}$$

and one sees that

$$\begin{aligned}\widetilde{K}_{a,b}(z_0, z_0'; \Gamma_1, \Gamma_2) &= \frac{1}{(b-1)!}\left(\widetilde{I}_{A^{-1}}(a+b, z_0, z_0', b; \Gamma_1) + A^{a-b+1}(\Gamma)\widetilde{I}_{A^{-1}}(a+b, z_0', z_0, a+1; \Gamma_2)\right)\\ &= \exp\left(-\frac{\overline{z_0}z_0'}{A}\right)\frac{1}{(b-1)!}\left(I_{A^{-1}}(a+b, z_0, z_0', b) + A^{a-b+1}I_{A^{-1}}(a+b, z_0', z_0, a+1)\right)\\ &= \exp\left(-\frac{\overline{z_0}z_0'}{A}\right)K^*_{a+b}(z_0, z_0', b) \qquad \text{by (2.35)}\end{aligned}$$

Hence,

$$\widetilde{K}_{a,b}(z_0, z_0'; \Gamma_1, \Gamma_2) = \exp\left(-\frac{\overline{z_0}z_0'}{A}\right)e^*_{a,b}(z_0, z_0') \tag{4.17}$$

On the other hand,

$$\begin{aligned}\Theta_{(z_0,z_0')}(z, z') &= \exp\left(-\frac{z_0\overline{z_0'}}{A}\right)\exp\left(-\frac{z\overline{z_0'} + z'\overline{z_0}}{A}\right)\Theta(z+z_0, z'+z_0')\\ &= \exp\left(\frac{(z+z_0)\overline{z'} + (z'+z_0')\overline{z'}}{A}\right)\exp\left(\frac{z_0\overline{z_0'}}{A}\right)\widetilde{K}_{0,1}(z+z_0, z'+z_0', \Gamma, \Gamma)\\ &= u_{z,z'}(\overline{z}, \overline{z'})\exp\left(\frac{z_0\overline{z_0'}}{A}\right)\widetilde{K}_{0,1}(z+z_0, z'+z_0', \Gamma, \Gamma)\end{aligned}$$

Now, $\widetilde{K}_{0,1}(z, z', \Gamma, \Gamma)$ has singularities in (z_0, z_0') when $\gamma = -z_0, -z_0'$. This means that the principal part of its Laurent series is given by

$$\frac{1}{z-z_0}\widetilde{I}_{A^{-1}}(1, z, z', 1; \{-z_0\}) + \frac{1}{z'-z_0'}\widetilde{I}_{A^{-1}}(1, z', z, 1; \{-z_0'\})$$

where

$$\begin{aligned}\widetilde{I}_{A^{-1}}(1, z+z_0, z'+z_0', 1; \{-z_0\}) &= \exp\left(-\frac{(\overline{z+z_0})(z'+z_0')}{A(\Gamma)}\right)\int_{A^{-1}(\Gamma)}^{\infty} \theta_{1,t}(z+z_0, z'+z_0'; \{-z_0\})\, dt \\ &= \exp\left(-\frac{(\overline{z+z_0})(z'+z_0')}{A(\Gamma)}\right)\overline{z}\langle\, z', z_0\,\rangle_\Gamma \int_{A^{-1}(\Gamma)}^{\infty} \exp\left(-t|z|^2\right)\, dt \\ &= \frac{1}{z}\exp\left(-\frac{(\overline{z+z_0})(z'+z_0')}{A(\Gamma)}\right)\exp\left(-\frac{z\overline{z}}{A(\Gamma)}\right)\langle\, z', z_0\,\rangle_\Gamma\end{aligned}$$

and similarly,

$$\widetilde{I}_{A^{-1}}(1, z'+z_0', z+z_0, 1; \{-z_0'\}) = \frac{1}{z'}\exp\left(-\frac{(\overline{z+z_0})(z'+z_0')}{A(\Gamma)}\right)\exp\left(-\frac{z'\overline{z'}}{A(\Gamma)}\right)\langle\, z, z_0\,\rangle_\Gamma.$$

Now, pose

$$\begin{cases} v_{z,z'}(\overline{z}, \overline{z'}) & := \exp\left(\frac{z_0\overline{z_0'}}{A(\Gamma)}\right)\widetilde{I}_{A^{-1}}(1, z+z_0, z+z_0, 1; \{-z_0\}) \\ v'_{z,z'}(\overline{z}, \overline{z'}) & := \exp\left(\frac{z_0'\overline{z_0}}{A(\Gamma)}\right)\widetilde{I}_{A^{-1}}(1, z'+z_0', z+z_0, 1; \{-z_0'\}) \end{cases}$$

These are real analytic functions in $(z, z', \overline{z}, \overline{z'})$, with $v_{z,z'}(0,0) = v'_{z,z'}(0,0) = 1$. Observe that whenever $z_0, z_0' \in \Gamma$, one has

$$\widetilde{I}_{A^{-1}}(1, z, z', 1; \Gamma) = \widetilde{I}_{A^{-1}}(1, z, z', 1; \{-z_0\}) + \widetilde{I}_{A^{-1}}(1, z, z', 1; \Gamma_1)$$

and

$$\widetilde{I}_{A^{-1}}(1, z', z, 1; \Gamma) = \widetilde{I}_{A^{-1}}(1, z', z, 1; \{-z_0'\}) + \widetilde{I}_{A^{-1}}(1, z', z, 1; \Gamma_2).$$

and this finally gives:

$$\begin{aligned}\Theta_{(z_0,z_0')}(z, z') - u_{z,z'}(\overline{z}, \overline{z'})v_{z,z'}(\overline{z}, \overline{z'})\langle\, z_0', z_0\,\rangle\frac{\delta(z_0,\Gamma)}{z} - u_{z,z'}(\overline{z}, \overline{z'})v_{z,z'}(\overline{z}, \overline{z'})\frac{\delta(z_0',\Gamma)}{z'} \\ = u_{z,z'}(\overline{z}, \overline{z'})\exp\left(\frac{z_0\overline{z_0'}}{A}\right)\widetilde{K}_{0,1}(z+z_0, z'+z_0', \Gamma_1, \Gamma_2)\end{aligned}$$

By the same reasoning as in (i), one finally gets

$$\begin{aligned}&\left(\frac{\partial}{\partial z}\right)^{b-1}\left(\frac{\partial}{\partial z'}\right)^{a}\left(\Theta_{(z_0,z_0')}(z, z') - \langle\, z_0', z_0\,\rangle\frac{\delta(z_0,\Gamma)}{z} - \frac{\delta(z_0',\Gamma)}{z'}\right) \\ &= (-1)^{a+b-1}\frac{(b-1)!}{A^a}\exp\left(\frac{\overline{z_0}z_0'}{A}\right)\widetilde{K}_{a,b}(z_0, z_0'; \Gamma_1, \Gamma_2)z^{b-1}z'^{a} \\ &= (-1)^{a+b-1}\frac{(b-1)!}{A^a}\exp\left(\frac{\overline{z_0}z_0'}{A}\right)\frac{e^*_{a,b}(z_0, z_0')}{a!A^a}z^{b-1}z'^{a} \qquad \text{by (4.17)}\end{aligned}$$

which finishes the proof.

□

5

Algebraicity of the Eisenstein-Kronecker numbers and Damerell's theorem

The algebraicity of special values of L-functions is a very useful arithmetic property. Two approaches are being explored here: the first approach, uses mainly results from chapter 2 to prove Damerell's theorem on the algebraicity of the special values of the Hecke L-function, where the second is, seemingly more interesting and powerful as it allows the study of theta functions (chapter 4) on general CM abelian varieties algebraically, through the heavy machinery of *algebraic theta functions* [DM91].

5.1. Algebraicity results using Eisenstein-Kronecker series

Damerell's theorem ([Dam70]) is related to the algebraicity of some special values of Hecke L-functions, on imaginary quadratic fields. The relevance and important of such a result arise from its link to critical values of L-functions of elliptic curves, under complex multiplication (theorem B.3.10 - theorem 3.2.12)

Theorem 5.1.1 (Damerell)**.** *Let K be an imaginary quadratic number field, $\mathcal{O}_K$ its ring of integers. Then for all integers $a, b \geq 0$ such that $b - a \geq 3$*

$$\mathcal{B}^*_{a,b}(\mathcal{O}_K) := (-1)^{b-a} \frac{(b-1)!}{2A^a \Omega^{b+a}} e^*_{a,b}(\mathcal{O}_K) \qquad \text{are algebraic over } \mathbb{Q}.$$

Where $\Omega := 2\pi |q|^{\frac{1}{12}} \prod_{n\geq 1} (a - q^n)^2$ with $q := e^{2\pi i \tau}$ and A is the area of the fundamental domain of $\mathcal{O}_K$ divided by π. In particular, we have

$$\mathcal{B}_{a,b}(\mathcal{O}_K) \in \mathbb{Q}\left(e_4(\Omega\mathcal{O}_K), e_6(\Omega\mathcal{O}_K)\right)$$

Proof. Let us first fix a complex embedding $i_\infty : K \hookrightarrow \mathbb{C}$ such that $i_\infty(\mathcal{O}_K)$ is a lattice in $\mathbb{C}$ along with a basis $\{1, \tau\}$ of $\mathcal{O}_K$ with

$$\Im(\tau) > 0 \quad \text{and} \quad A = \frac{\tau - \overline{\tau}}{2\pi i}.$$

Let $\Gamma = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ be a lattice in $\mathbb{C}$ such that $E(\mathbb{C}) \cong {}^{\mathbb{C}}/_{\Gamma}$ has CM by $\mathcal{O}_K$. Recall from remark 2.2.11 and from (2.33) that

$$e^*_{a,b}(\mathcal{O}_K) = \frac{A^a}{2^a(b-1)\dots(b-a)} Q_{a,b}\left(e^*_2(\mathcal{O}_K), e^*_4(\mathcal{O}_K), \dots, e^*_{a+b}(\mathcal{O}_K)\right)$$

with

$$Q_{a,b} \in \mathbb{Q}[e_2^*(\mathcal{O}_K), e_4^*(\mathcal{O}_K), \dots, e_{a+b}^*(\mathcal{O}_K)] \quad \text{and} \quad e_{a,b}^* = e_{a,b} \quad \text{whenever } b-a \geq 3.$$

It thus, suffices to prove that the $e_{2k}^*(\mathcal{O}_K)$ are algebraic over $\mathbb{Q}$ for all values of $n = 2k$, $2 \leq n \leq b+a$. Recall from (2.16) the power series expansion for the Eisenstein series E_n for all $n \geq 2$:

$$E_n(z;\Gamma) - \frac{1}{z^n} = (-1)^n \sum_{k=1}^{\infty} \binom{2k-1}{n-1} e_{2k}(\Gamma) z^{2k-n}$$

where, $e_{2k}(\Gamma)$ is the value of $E_{2k}(z,(\Gamma)) - \frac{1}{z^{2k}}$ near 0, given by

$$\begin{cases} e_n(\Gamma) = \sum_{m_1,m_2\neq 0}^{e} \frac{1}{(m_1\omega_1+m_2\omega_2)^{2k}} & \text{if } n = 2k \\ e_n(\Gamma) = 0 & \text{else.} \end{cases}$$

In particular, one has that

$$\begin{aligned} E_2(z,\omega_1,\omega_2) &= \frac{1}{z^2} + \sum_{k=1}^{\infty} e_{2k}(\Gamma) z^{2k-2} \\ E_4(z;\Gamma) &= \frac{1}{z^4} + \sum_{k=1}^{\infty} \frac{(2k-1)(2k-2)(2k-3)}{6} e_{2k}(\Gamma) z^{2k-4} \\ &= (E_2(z,\omega_1,\omega_2) - e_2(\Gamma))^2 - 5e_4(\Gamma) \qquad \text{by (2.25)} \end{aligned}$$

This gives a recursive formula

$$e_{2k}(\Gamma) = \sum_{i=2}^{k-2} 3\frac{(2i-1)(2k-2i-1)}{(k-3)(4k^2-1)} e_{2i}(\Gamma) e_{2(k-i)}(\Gamma) \in \mathbb{Q}[e_4(\Gamma), e_6(\Gamma)] \tag{5.1}$$

(i) For simplicity, we first suppose that $4 \leq n = 2k \leq b+a$, so we have that $e_{a,b}^*(\Gamma) = e_{a,b}(\Gamma)$.

- **Claim**$_1$: For any linear transformation Γ' of the lattice Γ such that

$$\Gamma' = \omega_1'\mathbb{Z} \oplus \omega_2'\mathbb{Z} \quad \text{where} \quad \begin{cases} \omega_1' &= a\omega_1 + b\omega_2 \\ \omega_2' &= c\omega_1 + d\omega_2 \end{cases} \quad \text{for } a,b,c,d \in \mathbb{Z}$$

One has:

$$e_{2k}' := e_{2k}(\Gamma') \in \mathbb{Q}(e_4(\Gamma), e_6(\Gamma)) \qquad \text{for all } n \geq 4. \tag{5.2}$$

Proof of Claim$_1$. Let $D := |ad-bc|$ such that $[\Gamma' : \Gamma] = D = [D\Gamma : \Gamma']$ and let R (resp. R') be a set of representatives for $\Gamma/_{D\Gamma}$ $\left(\text{resp. } \Gamma'/_{D\Gamma}\right)$ containing 0.

Recall from (2.20) that, for any $n \geq 2$*:

$$E'_n\left(\frac{z+r}{D};\Gamma'\right) = \sum_{r\in R} E_n\left(\frac{z+r}{D};\Gamma\right) = D^n E_n(z;\Gamma)$$

Hence, subtracting $\left(\frac{D}{z}\right)^n$ in both sides and taking $z = 0$, one gets

$$\sum_{\substack{r\in R\\ r\neq 0}} E_n\left(\frac{r}{D};\Gamma\right) = (D^n - 1)e_n(\Gamma)$$

Recall also that, for $n = 2k$:

$$\begin{aligned} e'_{2k} &= \sum_{r'\in R'}\sum_{\gamma\in\Gamma} \frac{1}{(r'+D\gamma)^{2k}} = \frac{e_{2k},\Gamma}{D^{2k}} + \frac{1}{D^{2k}} \sum_{\substack{r'\in R'\\ r'\neq 0}} \frac{1}{\left(\frac{r'}{D}+\gamma\right)^{2k}} \\ &= \frac{e_{2k},\Gamma}{D^{2k}} + \frac{1}{D^{2k}} \sum_{\substack{r'\in R'\\ r'\neq 0}} E_{2k}\left(\frac{r'}{D};\Gamma\right) \end{aligned}$$

Thus, any power of $E'_{2k}\left(\frac{z+r}{D};\Gamma'\right)$ is a polynomial in E_n and $\wp$. By Newton's algorithm : power sums $\leftrightarrow$ elementary symmetric polynomials, the same holds for the elementary symmetric polynomials in E_{2k} and thus, by Galois theory (or more precisely, by the fundamental theorem of symmetric polynomials) one gets

$$E_n\left(\frac{z+r}{D};\Gamma\right) \text{ and } \wp\left(\frac{z+r}{D};\Gamma\right) \text{ are algebraic over } \mathbb{Q}\left(\wp\left(\frac{z+r}{D};\Gamma\right), E_n\left(\frac{z+r}{D};\Gamma\right)\right).$$

Now, as

$$\begin{aligned} \sum_{\substack{r\in R\\ r\neq 0}} \wp\left(\frac{r}{D};\Gamma\right) &= \sum_{\substack{r\in R\\ r'\neq 0}} E_2\left(\frac{r}{D};\Gamma\right) - \sum_{\substack{r\in R\\ r'\neq 0}} e_2(\Gamma) \\ &= (D^2-1)e_2(\Gamma) - (D^2-1)e_2(\Gamma) = 0 \end{aligned}$$

One sees that for all $r' \in R' \subseteq R$, $E_{2k}\left(\frac{r'}{D};\Gamma\right)$ is algebraic over $\mathbb{Q}(e_{2k}(\Gamma))$. The claim follow from the recursive formula (5.1). ☐

Now, let $\alpha \in \mathbb{C}^\times$ and suppose that the linear transformation is given by complex multiplication, i.e. $\Gamma' = \omega'_1\mathbb{Z} \oplus \omega'_2\mathbb{Z} = \alpha\Gamma \subseteq \Gamma$, where:

$$\omega'_1 = \alpha\omega_1, \quad \omega'_2 = \alpha\omega_2 \quad \text{and} \quad D = |\alpha|^2 > 0.$$

*Note that for $n = 2$ and by remark 2.2.4, the following also holds:

$$\sum_{r\in R} \wp\left(\frac{z+r}{D}\right) = D^2\wp(z)$$

(note here that, for $\alpha\Gamma$ to be a sub-lattice of Γ, α has to be in $\mathcal{O}_K$ or in $\mathbb{Z}$.) Observe from §3.3.1 that, a lattice Γ' has CM by $\mathcal{O}_K$ if and only if

$$e_4(\Gamma') = \beta^{-4}e_4(\Gamma) \quad \text{and} \quad e_6(\Gamma') = \beta^{-4}e_6(\Gamma) \quad \text{for some } \beta \in \mathcal{O}_K.$$

Pose $\delta(\Gamma) := e_4(\Gamma)^3 e_6(\Gamma)^{-2}$, then

$$\Gamma' = \alpha\Gamma \quad \Leftrightarrow \quad \delta(\Gamma') = \delta(\Gamma)$$

- **Claim**$_2$: If Γ admits CM by $\mathcal{O}_K$, then $\delta(\Gamma)$ is algebraic over $\mathbb{Q}$.

Proof of Claim$_2$. For integers $a, b, c, d \in \mathbb{Z}$, define the function

$$T_4(u,v) := e_4(u'\mathbb{Z} \oplus v'\mathbb{Z}) - \alpha^{-4}e_4(u\mathbb{Z} \oplus v\mathbb{Z}) \quad \text{where} \quad \begin{cases} u' = & au + cv. \\ v' = & bu + dv. \end{cases}$$

Then T_4 cannot be identically 0, as $\alpha \notin \mathbb{Z}$. By **Claim**$_1$, $T_4(u,v)$ is then algebraic over $\mathbb{Q}(e_4(u\mathbb{Z}\oplus v\mathbb{Z}), e_6(u\mathbb{Z}\oplus v\mathbb{Z}))$ and there exists a non-zero polynomial $P \in \mathbb{Q}[T_4, e_4(u\mathbb{Z} \oplus v\mathbb{Z}), e_6(u\mathbb{Z} \oplus v\mathbb{Z})]$ such that

$$P(T_4(u,v), e_4(u\mathbb{Z} \oplus v\mathbb{Z}), e_6(u\mathbb{Z} \oplus v\mathbb{Z})) = 0$$

Let $P = T_4^m Q$ for some $m \geq 0$, where Q is a rational polynomial in the variables $T_4(u,v), e_4(u\mathbb{Z} \oplus v\mathbb{Z}), e_6(u\mathbb{Z} \oplus v\mathbb{Z})$ that is NOT a multiple of $T_4(u,v)$. By continuity, and since $T_4 \not\equiv 0$ one still has that

$$Q(T_4(u,v), e_4(u\mathbb{Z} \oplus v\mathbb{Z}), e_6(u\mathbb{Z} \oplus v\mathbb{Z})) = 0$$

However, since in our case $T_4(\omega_1, \omega_2) = 0$, we must have

$$Q(0, e_4(\Gamma), e_6(\Gamma)) = 0$$

Thus, by the homogeneity of $e_4(\Gamma)$ and $e_6(\Gamma)$, one gets

$$\begin{cases} \delta(\Gamma') \text{ is algebraic over } \mathbb{Q} & \text{if } e_6(\Gamma) \neq 0 \\ \delta(\Gamma') = \infty & \text{else.} \end{cases}$$

□

Now take $\Gamma = \mathcal{O}_K$ and $\alpha = \Omega := 2\pi|q|^{\frac{1}{12}} \prod_{n\geq 1}(a - q^n)^2$ with $q := e^{2\pi i\tau}$ (observe here that $\tau \in \mathbb{Q}(\Omega)$). Suppose at first, that $e_6(\mathcal{O}_K) \neq 0$, then from (3.10) :

$$\begin{aligned} \delta(\mathcal{O}_K) &= 60^3 e_4(\mathcal{O}_K)^3 - 3^3 140^2 e_6(\mathcal{O}_K)^2 \\ &= e_4(\mathcal{O}_K)^3 \left(60^3 - \frac{3^3 140^2}{e_4(\mathcal{O}_K)^3 e_6(\mathcal{O}_K)^{-2}}\right) \\ &= e_4(\mathcal{O}_K)^3 \left(60^3 - \frac{3^3 140^2}{\delta(\mathcal{O}_K)}\right) \\ &= e_6(\mathcal{O}_K)^2 \left(60^3 \delta(\mathcal{O}_K) - 3^3 140^2\right) \end{aligned}$$

Hence

$$(e_4')^3 := e_4(\Omega\mathcal{O}_K)^3 = \pm\frac{e_4(\mathcal{O}_K)^3}{\Delta(\mathcal{O}_K)} = \pm\left(60^3 - \frac{3^3 140^2}{\delta(\mathcal{O}_K)}\right)^{-1}$$

$$(e_6')^2 := e_6(\Omega\mathcal{O}_K)^2 = \pm\frac{e_6(\mathcal{O}_K)^2}{\Delta(\mathcal{O}_K)} = \pm\left(60^3(\delta(\mathcal{O}_K)) - 3^3 140^2\right)^{-1}.$$

As $\delta(\mathcal{O}_K)$ is algebraic over $\mathbb{Q}$ from $\mathbf{Claim}_2$, so are e_4' and e_6'.
Now if $e_6(\mathcal{O}_K) = 0$: This is known as the "*lemniscatic case*" and occurs when $\Omega = i$ (and thus $K = \mathbb{Q}(i)$), then the same holds since

$$e_4(\Omega\mathcal{O}_K)^3 = \Omega^{-12} e_4(\mathcal{O}_K)^3 = \pm 60^{-3}$$
$$e_6(\Omega\mathcal{O}_K) = 0$$

Hence, by $\mathbf{Claim}_1$, $e_4(\mathcal{O}_K), e_6(\mathcal{O}_K), \ldots, e_{a+b}(\mathcal{O}_K)$ are all algebraic over $\mathbb{Q}$.

(ii) We now treat similarly the case $n = 2$, where $e_2^*(\mathcal{O}_K) = e_2 - \frac{1}{A}$.

Let $\alpha \in \mathcal{O}_K \smallsetminus \mathbb{Z}$ and $\mathfrak{a} \lhd \mathcal{O}_K$ with $\mathfrak{a} = (\alpha)$. Let R be a set of representatives for $\mathcal{O}_K/_{\mathfrak{a}}$ containing 0. Recall that by definition, from §2.2.9

$$\sum_{r\in R} E_2^*(z+r;\mathcal{O}_K) = \alpha^{-2}\sum_{r\in R} E_2^*\left(\frac{z+r}{\alpha};\mathcal{O}_K\right) = E_2^*(z,\mathcal{O}_K)$$

$$\Rightarrow \quad \alpha^{-2}\sum_{r\in R} E_2^*\left(\frac{z+r}{\alpha};\mathcal{O}_K\right) - \frac{1}{z^2} = \alpha^{-2}\left(\sum_{r\in R} E_2^*\left(\frac{z+r}{\alpha};\mathcal{O}_K\right) - \frac{\alpha^2}{z^2}\right)$$
$$= E_2^*(z,\mathcal{O}_K) - \frac{1}{z^2}$$

Now, taking $z = 0$:

$$\sum_{\substack{r\in R\\ r\neq 0}} E_2^*\left(\frac{r}{\alpha};\mathcal{O}_K\right) = \alpha^2 e_2^*(\mathcal{O}_K)$$

$$\Rightarrow \quad \sum_{\substack{r\in R\\ r\neq 0}} \left(E_2^*\left(\frac{z+r}{\alpha};\mathcal{O}_K\right) - e_2^*(\mathcal{O}_K)\right) = (\alpha^2 - |\alpha|^2) e_2^*(\mathcal{O}_K)$$

$$\Rightarrow \quad \sum_{\substack{r\in R\\ r\neq 0}} \wp\left(\frac{r}{\alpha};\mathcal{O}_K\right) = \alpha(\alpha - \overline{\alpha}) e_2^*(\mathcal{O}_K)$$

Hence, by the same argument, one shows that $\wp\left(\frac{\overline{\alpha}r}{|\alpha|^2};\mathcal{O}_K\right)$ is algebraic over $\mathbb{Q}(e_4(\mathcal{O}_K), e_6(\mathcal{O}_K))$ and thus, so is $e_2^*(\mathcal{O}_K)$. This finishes the proof.

□

Remark 5.1.2. Using the analytic continuation of the Eisenstein-Kronecker-Lerch series (as done in the last part of §2.3.6 where, $e_{a,b}^* = K_{a+b}^*(0,0,b)$), Damerell's theorem is valid for all values $b > a \geq 0$.

5.2. Using the theory of algebraic theta functions

The idea is the following: Let X be a g-dimensional abelian variety over some ground field $\mathbb{F}$ with $X(\mathbb{C}) \cong \mathbb{T} = {}^{V}\!/_{\mathbb{C}}$ (as explained in B.1). Then for a given sections s, one can use the translation operator of reduced theta functions to reduce the study of properties of ϑ_s at a torsion point $p \in \Gamma \otimes \mathbb{Q}$ to the study of ϑ at the origin. Mumford's theory provides a way to construct an *algebraic* translation operator that preserves the reducedness and algebraicity.

5.2.1. Review of Mumford's theory of algebraic theta functions

Let $A(\mathbb{F})$ be a g-dimensional abelian variety over a ground field $\mathbb{F}$. As seen in the previous chapter, classical theta functions arise by trivialising a complex line bundle on the universal covering of an abelian variety. Unfortunately, in the "Algebraic Geometry" world, and over an arbitrary ground field $\mathbb{F}$, there is no universal cover in the category of varieties; meaning that the universal covering is not *algebraic*. However, there are many finite algebraic coverings in-between: One noticeable example is given by the N-multiplication map $[N]$. These finite coverings form a projective system:

Definition 5.2.2 (Tate module)**.** Let $l \neq \operatorname{char}\mathbb{F}$ be a prime number, and $A(\mathbb{F})$ be an abelian variety. We define the **l-adic Tate module** associated to $A(\mathbb{F})$ to be the $\mathbb{Z}_l$-module

$$T_l(A) \coloneqq \varprojlim_{n\to\infty} A[l^n]$$

where, for all $x \in T_l(A)$, $x = (a_i)_{i\geq 0}$ for some torsion points a_i of $A(\overline{\mathbb{F}})$ such that: for all $i \geq 0$

$$a_i \in A[l^i] \quad \text{and} \quad l \cdot a_{i+1} = a_i.$$

Note that one has a sequence of groups

$$\dots \xrightarrow{[l]} A[l^{n+1}] \xrightarrow{[l]} A[l^n] \xrightarrow{[l]} \dots \xrightarrow{[l]} A[l] \xrightarrow{[l]} 0.$$

Remark 5.2.3. If $\mathbb{F} \subset \mathbb{C}$, the complex analytic structure on $A(\mathbb{C}) \cong {}^{\mathbb{C}^g}\!/_{\Gamma}$ provides a $\mathbb{Z}$-module structure on Γ along with a complex structure $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} (\cong T_0 A)$. The multiplication map $[N]$ induces a canonical isomorphism

$$A[N] \cong \Gamma \otimes_{\mathbb{Z}} \left({}^{\mathbb{Z}}\!/_{N\mathbb{Z}}\right)$$

and, using (B.2), one gets

$$\dots \xrightarrow{\text{red mod } l^{n+1}} \Gamma \otimes_{\mathbb{Z}} \left({}^{\mathbb{Z}}\!/_{l^{n+1}\mathbb{Z}}\right) \xrightarrow{\text{red mod } l^n} \Gamma \otimes_{\mathbb{Z}} \left({}^{\mathbb{Z}}\!/_{l^{n+1}\mathbb{Z}}\right) \xrightarrow{\text{red mod } l^{n-1}} \dots \longrightarrow 0$$

Hence, taking the inverse limit, one has a (non-canonical) isomorphism

$$T_l(A) \cong \Gamma \otimes_{\mathbb{Z}} \mathbb{Z}_l^{2g}$$

Definition 5.2.4 (Adèlic Tate module)**.** Let $l \neq \operatorname{char}\mathbb{F}$ be a prime number, and $A(\mathbb{F})$ be an abelian variety. Adèlically, one defines the Tate module associated to $A(\mathbb{F})$ to be the inverse limit of the system of N-torsion points along the transition maps, given by

$$\widehat{T}(A) := \prod_{l \neq \operatorname{char}\mathbb{F}} T_l(A) = \{x_l \in \prod_{l \neq \operatorname{char}\mathbb{F}} T_l(A) \text{ such that all but finitely } x_l \in T_l(A)\}$$

Hence, the adèlic Tate module may be regarded as an *algebraic* version of the complex universal cover of $A(\mathbb{F})$. Consider a point $a = (a_i)_i \in \widehat{T}(A)$ and let $n = 2kN$, where N is a non zero integer such that $N \cdot a_1 = 0$, $k \in \mathbb{N}_{>0}$.

Let $\mathcal{L}$ be a symmetric line bundle on $A(\mathbb{F})$ i.e. there exists an isomorphism

$$\rho_{-1} : [-1]^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$$

In order to compare sections of $\mathcal{L}$ at the torsion point a in $A(\overline{\mathbb{F}})$ and sections of $\mathcal{L}$ at the origin, one needs to construct a -somehow- canonical isomorphism between $\mathcal{L}$ and $t_a^*\mathcal{L}$. The difficulty here, is that this translation operator has to be compatible with the system of points in $\widetilde{T}(A)$, as it is not translating a point of A but of its algebraic cover.

Mumford proceeds as follow: Let $b = (bi)_i \in \widehat{T}(A)$ such that $a = 2b$ (so $b_i = a_{2i}$) and pose $\mathcal{L}_{b_n} := t_{b_n}^*([n]^*\mathcal{L}) \otimes ([n]^*\mathcal{L})^{-1}$.

First, observe that

$$\begin{aligned}
\mathcal{L}_{b_n} &= t_{b_n}^*([n]^*\mathcal{L}) \otimes ([n]^*\mathcal{L})^{-1} & \\
&= [n]^*\left(t_{b_1}^*\mathcal{L} \otimes \mathcal{L}^{-1}\right) & \text{by pulling back } [n]^* \\
&\cong t_{2kNb_1}^*\mathcal{L} \otimes \mathcal{L}^{-1} & \text{by theorem B.2.6} \\
&\cong \mathscr{O}_A. &
\end{aligned}$$

Fix such an isomorphism and call it

$$\rho_n : \mathcal{L}_{b_n} \xrightarrow{\sim} \mathscr{O}_A.$$

One sees that

$$[-1]^*\mathcal{L}_{b_n} = t_{-b_n}^*([-n]^*\mathcal{L}) \otimes [-n]^*\mathcal{L}^{-1} \xrightarrow[\sim]{\rho_{-1}\otimes\rho_{-1}^{\otimes -1}} t_{-b_n}^*([n]^*\mathcal{L}) \otimes ([n]^*\mathcal{L})^{-1} \tag{5.3}$$

And

$$\begin{aligned}
[n]^*(t_{a_1}^*\mathcal{L}) \otimes ([n]^*\mathcal{L})^{-1} &= t_{a_n}^*([n]^*\mathcal{L}) \otimes ([n]^*\mathcal{L})^{-1} \\
&= t_{b_n}^*\left(t_{b_n}^*([n]^*\mathcal{L}) \right) \otimes t_{b_n}^*\left(t_{-b_n}^*([n]^*\mathcal{L})^{-1}\right) \\
&= t_{b_n}^*\left(t_{b_n}^*([n]^*\mathcal{L}) \otimes t_{-b_n}^*([n]^*\mathcal{L})^{-1} \right) \\
&= t_{b_n}^*\left(t_{b_n}^*([n]^*\mathcal{L}) \otimes ([n]^*\mathcal{L})^{-1} \otimes ([n]^*\mathcal{L}) \otimes t_{-b_n}^*([n]^*\mathcal{L})^{-1} \right) \\
\text{(from (5.3))} \quad &\cong t_{b_n}^*\left(\mathcal{L}_{b_n} \otimes [-1]^*\mathcal{L}_{b_n}^{-1}\right) \xrightarrow[\sim]{\rho_n\otimes\rho_n^{\otimes -1}} t_{b_n}^*\mathscr{O}_A \cong \mathscr{O}_A
\end{aligned} \tag{5.4}$$

Hence, one has a canonical isomorphism

$$[n]^*t_a^{\mathcal{M}} : [n]^*(t_{a_1}^*\mathcal{L}) \xrightarrow{\sim} [n]^*\mathcal{L}$$

Since, say for $k = 1$: $4N^2 \cdot b_{2N} = N \cdot 4N \cdot a_{4N} = N \cdot a_1 = 0$ (as $a_1 \in A[N]$), b_{2N} is defined over $\mathbb{F}(A[4N^2])$ and thus, so is $[n]^*t_{a_1}^{\mathcal{M}}$.

Remark 5.2.5. Note that the isomorphisms in (5.3) and (5.4) DO NOT depend on the choice of ρ^{-1} resp. ρ_n. Indeed, since the automorphisms of a line bundle are multiplications by a scalar, it does not matter which scalar one takes as it will always cancel with its inverse in $\rho_{-1} \otimes \rho_{-1}^{\otimes -1}$ (resp. $\rho_n \otimes \rho_n^{\otimes -1}$).

Geometrically, the trivialisation (5.4) above produces a rational section

$$\begin{aligned} \mathcal{O}_A &\xrightarrow{\sim} [n]^*(t_{a_1}^*\mathcal{L}) \otimes ([n]^*\mathcal{L})^{-1} \\ 1 &\longmapsto f_n^{\mathcal{M}} \end{aligned}$$

If one considers a Cartier divisor $D = (U_i, f_i)_i$, with its corresponding invertible sheaf $\mathcal{L} = \mathcal{O}_A(D)$, then this geometric isomorphism is given on $V_i = n^{-1}(U_i)$ by the gluing data:

$$\begin{tikzcd}
\operatorname{Spec}\operatorname{Sym}_{\mathcal{O}_{V_i}}\left(n^*t_{a_1}^*\mathcal{O}_A(-D)\right) \arrow[rrr] \arrow[ddd] & & & \operatorname{Spec}\operatorname{Sym}_{\mathcal{O}_{V_i}}\left(n^*\mathcal{O}_A(-D)\right) \arrow[ddd] \\
& ([n]^*f_i) \cdot f_n^{\mathcal{M}} & {[n]^*f_i} \arrow[l, mapsto] & \\
& \frac{[n]^*f_i}{[n]^*t_{a_1}^*f_i} \cdot f_n^{\mathcal{M}} \cdot Y_i \arrow[u, mapsto] & X_i \arrow[l, mapsto] \arrow[u, mapsto] & \\
\operatorname{Spec}\mathcal{O}_{V_i}[Y_i] \arrow[rrr] & & & \operatorname{Spec}\mathcal{O}_{V_i}[X_i]
\end{tikzcd}$$

In other words, one has the following:

Proposition 5.2.6. *Let $A(\mathbb{F})$ be an abelian variety over $\mathbb{F}$, $a = (a_i)_i \in \widehat{T}(A)$ and $n = 2kN$, where N is a non-zero integer such that $N \cdot a_1 = 0$, $k \in \mathbb{N}_{>0}$. There exist canonical rational functions $(f_n^{\mathcal{M}})_n$ defined over $\mathbb{F}(A[4N^2])$ such that*

$$D^{\mathcal{M}} := \operatorname{div}(f_n^{\mathcal{M}}) = [n]^*t_{a_1}^*(D) - [n]^*(D) \qquad \textit{where } D = (U_i, f_i)_i,$$

and the isomorphism $[n]^*t_a^{\mathcal{M}} : [n]^*(t_{a_1}^*\mathcal{L}) \xrightarrow{\sim} [n]^*\mathcal{L}$ is geometrically given by the data:

$$\begin{aligned} \mathbb{A}^1_{n^{-1}(U_i)} &\longrightarrow \mathbb{A}^1_{n^{-1}(U_i)} \\ h_i &\longmapsto (f_n^{\mathcal{M}}) \cdot \left(\frac{[n]^*f_i}{[n]^*t_{a_1}^*f_i}\right) \cdot g_i \end{aligned}$$

The next logical thing to do, is to trivialize a line bundle on $\widehat{T}(A)$, and regard the sections of this line bundle as the algebraic theta functions. This means that one would need to fix a projective system of trivialization:

Definition 5.2.7. Let $d \in \mathbb{N}$, $n = dk$ with $k \in \mathbb{N}_{>0}$ and let $(A_n)_n$ be a projective system in the category of schemes over $\mathbb{F}$ such that the following diagram commutes for all $m, n \in \mathbb{N}$:

$$\begin{tikzcd} X_{mn} \arrow[r] \arrow[d, "\pi_{mn}"'] & X_n \arrow[d, "\pi_n"] \\ A \arrow[r, "{[m]}"] & A \end{tikzcd} \qquad \text{and} \qquad \text{where } \pi_i \text{ are the natural projections.}$$

A system of trivialization of a line bundle $\mathcal{L}$, is a pair (X_n, φ_n) where $(X_n)_n$ is a projective system as above, and

$$\varphi_n : \pi_n^*[n]^*\mathcal{L} \cong \mathcal{O}_{A_n}$$

is an isomorphism of invertible sheaves, compatibles with the natural projections:

$$\begin{tikzcd} \mathcal{O}_{X_n} \cong \pi_n^*[n]^*\mathcal{L} \arrow[r] \arrow[d] & \dots \arrow[r] & {[n]^*\mathcal{L}} \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\ X_n \arrow[r, "\pi_n"] & \dots \arrow[r] & A \arrow[r, "{[n]}"] & A \end{tikzcd}$$

Example 5.2.8. Consider a point $a = (a_i)_i \in \widehat{T}(A)$ with $N \cdot a_1 = 0$ as before. One can construct a system of trivialisations from a as follow: Since $a_n \in A(\overline{\mathbb{F}})$, one has a morphism $\pi_{nN} : \mathrm{Spec}(\overline{\mathbb{F}}) \to A$. Fixing an isomorphism $[0]^*\mathcal{L} \cong \mathcal{O}_A$ induces a trivialisation $[n]^*\mathcal{L} \times_A A[n] \cong \mathcal{O}_{A[n]}$ as the following diagram shows:

$$\begin{tikzcd} & {[0]^*\mathcal{L} \times_{\overline{\mathbb{F}}} A[n]} \arrow[rr] \arrow[d] \arrow[dl, equal] & & {[0]^*\mathcal{L}} \arrow[d] & \\ {[n]^*\mathcal{L} \times_A A[n]} \arrow[rr] \arrow[dd] & A[n] \arrow[rr] \arrow[ddl] & {[n]^*\mathcal{L}} \arrow[dd] & \mathrm{Spec}(\overline{\mathbb{F}}) = X_{Nn} \arrow[r] \arrow[ddl, "\pi_{nN}"'] & X_N \arrow[ddl, "\pi_N"] \\ & & & & \\ A[n] \arrow[rr] & & A \arrow[r, "{[n]}"'] & A & \end{tikzcd}$$

Hence, one constructs a system of trivialization $(X_n, \varphi_{a,n})_{n \in N\mathbb{N}}$ with $\varphi_{a,n} : \pi_n^*[n]^*\mathcal{L} \cong \mathcal{O}_{X_n}$ for all $n \in N\mathbb{N}$.

Now, given a line bundle $\mathcal{L}(D)$ over $\widehat{T}(A)$, and given a system of trivialisations $(X_n, \varphi_{a,n})_{n \in d\mathbb{N}}$, one can obtain a system of trivialisations of $t_{a_1}^*\mathcal{L}$ by using the translation operator $t_a^{\mathcal{M}}$ as follow: let $d' = \mathrm{lcm}(d, 2N)$, then the isomorphism

$$\varphi_{a,n}^{\mathcal{M}} : \pi_n^*[n]^*t_{a_1}^*\mathcal{L} \xrightarrow[\sim]{\pi_n^*[n]^*t_a^{\mathcal{M}}} \pi_n^*[n]^*\mathcal{L} \xrightarrow[\sim]{\varphi_{a,n}} \mathcal{O}_{X_n}$$

is compatible with the natural projections and provides a system of trivialization $(X_n, \varphi_{a,n}^{\mathcal{M}})_{n \in d'\mathbb{N}}$ of $\mathcal{L}$ over $\widehat{T}(A)$.

Given a rational section $s = s_D$ of such a line bundle (corresponding to the Cartier divisor $D = (U_i, f_i)_i$), define the rational morphisms

$$[n]^*\vartheta_s : X_n \xrightarrow{\pi_n^*[n]^* s_D} \pi_n^*[n]^*\mathcal{L} \xrightarrow[\sim]{\varphi_{a,n}} \mathscr{O}_{X_n} \tag{5.5}$$

$$[n]^*\overset{\mathcal{M}}{\vartheta}{}_s^{+a} : X_n \xrightarrow{\pi_n^*[n]^* t_{a_1}^* s} \pi_n^*[n]^* t_{a_1}^*\mathcal{L} \xrightarrow[\sim]{\varphi_{a,n}^{\mathcal{M}}} \mathscr{O}_{X_n} \tag{5.6}$$

5.2.9. The "Majin" Kronecker theta function $\overset{\mathcal{M}}{\Theta}$

Let $A(\mathbb{F})$ be a g-dimensional abelian variety over $\mathbb{F}$, and suppose that $\mathbb{F}$ is a sub-field of $\mathbb{C}$. Given an algebraic divisor D, the goal now would be to relate Mumford's algebraic construction of theta functions to ϑ_D associated to a meromorphic section of a line bundle of the form $\mathcal{L}(H,\chi)$, satisfying the transformation formula (4.4). Recall that from §4.1.10, any meromorphic section s_D of $\mathcal{L}(H,\chi)$ induces a reduced theta function corresponding to a section $t_w^* s_D$ on $\mathcal{L}(H,\chi\cdot\alpha_v)$. This would determine ϑ_D^{+w} corresponding to the translated section $t_w^* s_D$ up to a complex scalar. Using proposition 5.2.6, one can construct a more convenient translation operator: Consider the function

$$\begin{aligned} e^{\mathcal{M}} &: V \times V \longrightarrow \mathbb{C}^\times \\ &(v,w) \longmapsto e_w^{\mathcal{M}}(v) := \exp\left(\pi H(v,w) + \frac{\pi}{2} H(w,w)\right) \end{aligned}$$

and define the *algebraic* analogous of the translation (4.6) to be

$$\overset{\mathcal{M}}{\vartheta}{}_D^{+w}(v) := (e_w^{\mathcal{M}})^{-1}(v)\vartheta_D(v+w)$$

One easily checks (as in §4.1.10) that $\overset{\mathcal{M}}{\vartheta}{}_D^{+w}$ is a reduced theta function for $\mathcal{L}(H,\chi\cdot\alpha_v)$. Moreover, unlike the previous translation ϑ_D^{+w}, this algebraic translation does not depend on $\widetilde{\chi}$.

Fix a complex polarization $\pi : A(\mathbb{C}) \cong \mathbb{T} = {}^{V}\!/_{\mathbb{C}}$ and consider a point $v \in \Gamma \otimes \mathbb{Q}$ corresponding to a torsion point of $A(\mathbb{C})$. As seen in appendix B, one has a map

$$\iota : \Gamma \otimes \mathbb{Q} \hookrightarrow A(\mathbb{C})_{\mathrm{tors}}$$

which extends to a map

$$\begin{aligned} \widetilde{\iota} : \Gamma \otimes \mathbb{Q} &\longrightarrow \widehat{T}(A) \\ v &\longmapsto a = (a_n)_n \text{ where } a_n = \iota\left(\frac{v}{n}\right) \end{aligned}$$

Let ϑ_D be a reduced theta function associated to a meromorphic section of a line bundle $\mathcal{L}(H,\chi)$ with divisor D. Fix $w \in \Gamma \otimes \mathbb{Q}$ with $a = (a_n)_n = \widetilde{\iota}(w)$, and $w' = \frac{w}{2}$ where

$b = (b_n)_n = \widetilde{\iota}(w')$ and $n \in \mathbb{N}$ such that $nw \in 2\Gamma$.

Recall that, from remark 5.2.5, the function $f_n^{\mathcal{M}}$ is independent of the choice of ρ_{-1} and ρ_n. Thus, one can choose

$$\rho_{-1} : v \longmapsto \frac{\vartheta_D(-v)}{\vartheta_D(v)} \qquad \text{and} \qquad \rho_n : v \longmapsto \exp\left(-\pi H(nv, w')\right) \cdot \frac{\vartheta_D(nv + w')}{\vartheta_D(nv)}$$

These functions are meromorphic, Γ-periodic (note that since H is a hermitian form, the exponential part is holomorphic in v, and the transformation formula (4.4) shows that it is Γ-periodic as long as $nw' \in \Gamma$) and have divisors $[-1]^*D - D$ and $t_{b_n}^*[n]^*D - [n]^*D$, respectively. Choose an isomorphisms

$$[-1]^*\mathcal{L} \xrightarrow[\sim]{\rho_{-1}} \mathcal{L} \qquad \text{and} \qquad \mathcal{L}_{b_n} \xrightarrow[\sim]{\rho_n} \mathscr{O}_A$$

to be defined, respectively, by $\rho_{-1}(v)$ and $\rho_n(v)$. By proposition 5.2.6, one has

$$\begin{aligned}
f_n^{\mathcal{M}}(v) &= t_{b_n}^*\left(\frac{\rho_n(v) \cdot t_{-b_n}^*\rho_{-1}(nv)}{\rho_n(-v) \cdot \rho_{-1}(nv)}\right) \\
&= \exp\left[-2\pi H(nv, w') - \pi H(w', w')\right] \frac{\rho_n(v + \frac{w'}{n}) \cdot \rho_{-1}(nv)}{\rho_n(-v - \frac{w'}{n}) \cdot \rho_{-1}(nv + w')} \\
&= \exp\left[-\pi H(nv, w) - \frac{\pi}{2} H(w, w)\right] \frac{\vartheta_D(-nv - w')}{\vartheta_D(nv + w')} \cdot \frac{\rho_{-1}(nv)}{\vartheta_D(-nv)} \cdot \frac{\vartheta_D(nv + w)}{\rho_{-1}(nv + w')} \\
&= e_w^{\mathcal{M}}(nv)^{-1} \cdot \rho_{-1}(nv + w') \cdot \frac{1}{\vartheta_D(nv)} \cdot \frac{\vartheta_D(nv + w)}{\rho_{-1}(nv + w')} \\
&= e_w^{\mathcal{M}}(nv)^{-1} \frac{\vartheta_D(nv + w)}{\vartheta_D(nv)} = \frac{[n]^* \overset{\mathcal{M}}{\vartheta}{}_D^{+w}(v)}{[n]^*\vartheta_D(v)}.
\end{aligned} \tag{5.7}$$

Now, define a system of trivialisations of $\mathcal{L}(H, \chi)$: Let $(A_n, \varphi_n)_{n \in \mathbb{N}}$ be a system of trivialisations in the category of complex varieties where $A_n = V$ and $\varphi_n = [n]^*\varphi_{\vartheta,D}$ where the geometric isomorphism $\pi^*\mathcal{L} \xrightarrow[\sim]{\varphi_{\vartheta,D}} \pi^*\mathcal{L}(H, \chi)$ is given by the gluing data:

$$\begin{aligned}
\mathbb{A}^1_{U_i} &\longrightarrow \mathbb{A}^1_{\pi^{-1}(U_i)} \\
g &\longmapsto \vartheta_D(v) \cdot \left(\frac{1}{f_i}\right) \cdot g_i
\end{aligned}$$

Then, the rational morphism in (5.5) corresponding to the section $[n]^*s_D$ satisfy

$$\begin{tikzcd}
V \arrow[r, "\pi_n^*[n]^*s_D"] \arrow[dr, "{[n]^*\varphi_{\vartheta,D}}"'] & \pi_n^*[n]^*\mathcal{L} \arrow[d, "\varphi_n"] \\
& \mathscr{O}_V
\end{tikzcd} \qquad \Rightarrow \qquad [n]^*\vartheta_s(v) = [n]^*\vartheta_D(v)$$

similarly, the rational *translation* morphism (5.6) corresponding to the section $[n]^*t_{a_1}^*s_D$ satisfy

$$
\begin{tikzcd}
V \arrow[r, "\pi_n^*[n]^*t_{a_1}^*s"] \arrow[dr, "\overset{\mathcal{M}}{\vartheta}{}_D^{+w}(v)"'] & \pi_n^*[n]^*t_{a_1}^*\mathcal{L} \arrow[d, "{[n]^*\varphi_{a,n}^{\mathcal{M}}}"] \\
 & \mathscr{O}_V
\end{tikzcd}
\quad \Rightarrow \quad
\begin{aligned}
[n]^*\overset{\mathcal{M}}{\vartheta}{}_s^{+w}(v) &= [n]^*\vartheta_s(v)\cdot\pi_n^*f_n^{\mathcal{M}}(v) \\
\text{by (5.7)} \quad &= [n]^*\vartheta_D(v)\cdot[n]^*\overset{\mathcal{M}}{\vartheta}{}_D^{+w}(v)\cdot[n]^*\vartheta_D(v)^{-1} \\
&= [n]^*\overset{\mathcal{M}}{\vartheta}{}_D^{+w}(v)
\end{aligned}
$$

Hence, given a complex analytic system of trivialisations $(A_n = V, \varphi_n = [n]^*\varphi_{\vartheta,D})_{n\in\mathbb{N}}$ of $\mathcal{L}$, with $\pi^*\mathcal{L} \xrightarrow[\sim]{\varphi_{\vartheta,D}} \pi^*\mathcal{L}(H,\chi)$, one has a system of trivialisations $(A_n = V, \varphi_{a,n}^{\mathcal{M}} = [n]^*\varphi_{\vartheta_D^{+w}}^{\mathcal{M}})_{n\in 2N\mathbb{N}}$ of $t_w^*\mathcal{L}$ coming from $a = (a_n)_n = \widetilde{\iota}(w) \in \widehat{T}(A)$ with $\pi^*t_w^*\mathcal{L} \xrightarrow[\sim]{\varphi} {}_{\vartheta_D^{+w}}\ \pi^*\mathcal{L}(H,\chi\cdot\alpha_w)$.
This induces an isomorphism

$$[n]^*t_w^*\mathcal{L} \cong [n]^*\mathcal{L}(H,\chi\cdot\alpha_w)$$

as $\overset{\mathcal{M}}{\vartheta}{}_D^{+w}(v)$ is the reduced theta function corresponding to $\mathcal{L}(H,\chi\cdot\alpha_w)$.

Now fix once and for all an isomorphism $t_w^*\mathcal{L} \cong \mathcal{L}(H,\chi\cdot\alpha_w)$ so that the section t_w^*s has (under this isomorphism) the corresponding reduced theta function

$$\overset{\mathcal{M}}{\vartheta}{}_s^{+w}(v) = f_n^{\mathcal{M}}\left(\frac{v}{n}\right)\vartheta_s(v).$$

Note that $\overset{\mathcal{M}}{\vartheta}{}_s^{+w}$ differs from ϑ_s only by an $2n^2$-root of unity, as the $(f_n^{\mathcal{M}})_{2N\mathbb{N}}$ are rational functions defined over $\mathbb{F}(A[2N^2])$.

Finally, in the same way as in (4.13), we define the *algebraic translation* of the Kronecker theta function by

$$\overset{\mathcal{M}}{\Theta}_{(w,w')}(z,z') := \overset{\mathcal{M}}{\Theta}(z,z')^{+(w,w')} \tag{5.8}$$

5.2.10. Application: the case of a CM Elliptic curve.

Let $K \subset \mathbb{F} \subset \mathbb{C}$ be an imaginary quadratic field, $E(\mathbb{F})$ an elliptic curve with CM by $\mathcal{O}_K$. Fix a Weierstrass model over $\mathcal{O}_K$:

$$Y^2 = 4X^3 - g_2X - g_3 \qquad g_2, g_3 \in \mathbb{F}$$

where Γ is the period lattice of the invariant differential $\omega = \frac{dx}{y}$, along with a complex uniformisation

$$
\begin{aligned}
\mathbb{C}/_{\Gamma} &\xrightarrow{\sim} E(\mathbb{C}) \\
z &\longmapsto (\wp(z;\Gamma), \wp'(z;\Gamma)).
\end{aligned}
$$

Recall from §4.2.5 that for $\mathcal{L}$ being the line bundle corresponding to the divisor $[0] \in \mathrm{Div}(E)$, one has a reduced theta function associated to $[0]$ given by

$$\theta(z) = \exp\left(-\frac{e_2^*(\Gamma)}{2}z^2\right)\sigma(z)$$

where, the Taylor expansion around $z = 0$is given by (A.1)

$$\sigma(z) = z - \frac{g_2}{2^4\cdot 3\cdot 5}z^5 - \frac{g_3}{2^3\cdot 3\cdot 5\cdot 7}z^7 + \dots$$

Hence, one deduces the Taylor expansion of θ around $z = 0$:

$$\theta(z) = z - \frac{e_2^*}{2}z^2 + \text{ higher terms of } e_2^*,\ g_2 \text{ and } g_3$$

From (ii) in the proof of Theorem 5.1.1, we showed that e_2^* was algebraic, and thus, as $g_2, g_3 \in \mathbb{F}$ the Taylor expansion of θ around 0 has its coefficients in $\mathbb{F}$. In fact, this shows that the coefficients of the Laurent series of the Weierstrass functions ζ, $\wp$ and $\wp'$ has coefficients in $\mathbb{F}$. In particular, by (4.10) one has:

Lemma 5.2.11. *The Laurent expansion of the Kronecker theta series $\Theta(z,z')$ at the origin has algebraic coefficients.*

Now, consider $A(\mathbb{F}) := E(\mathbb{F})\times E(\mathbb{F})$ and let $\mathcal{P}_E$ be the poincaré bundle associated to $E(\mathbb{F})$. Let s_D be the meromorphic section of $\mathcal{P}_E$ associated to the divisor $D = \Delta - (O\times E) - (E\times O)$ defined over $\mathbb{F}$, which corresponds to the reduced theta function

$$\Theta(z,z') = \frac{\vartheta_s(z+z')}{\vartheta_s(z)\vartheta_s(z')}$$

We just saw that the two variable reduced theta function Θ has its Laurent expansion's coefficients in $\mathbb{F}$. As it is clearly not a rational function, consider $w, w' \in \Gamma\otimes\mathbb{Q}$ and an integer $N > 0$ such that $Nw, Nw' \in \Gamma$. For $n \in 2N\mathbb{N}$, the rational function $f_n^{\mathcal{M}}$ is defined over $\mathbb{F}(E(2n^2))$, hence the Taylor expansion near 0 of the function

$$\overset{\mathcal{M}}{\vartheta_s^{+w}}(v) = f_n^{\mathcal{M}}\left(\frac{v}{n}\right)\vartheta_s(v)$$

has coefficients in $\mathbb{F}(E(2n^2))$. Similarly, $\overset{\mathcal{M}}{\vartheta_s^{+w'}}(v) = f_n^{\mathcal{M}}\left(\frac{v}{n}\right)\vartheta_s(v)$ has also coefficients in $\mathbb{F}(E(2n^2))$. Moreover, explicit computations using (4.14) show that

$$\Theta_{(w,w')}(z,z') = \underbrace{\exp\left(\frac{w'\overline{w} - w\overline{w'}}{2A}\right)}_{\in\mathbb{S}^1}\overset{\mathcal{M}}{\Theta}_{(w,w')}(z,z') \quad \text{for all } w, w' \in \Gamma\otimes\mathbb{Q}. \tag{5.9}$$

We finally obtain the following corollary:

Corollary 5.2.12. *Let E be an elliptic curve over $\mathbb{F} \subset \mathbb{C}$ with CM by $\mathcal{O}_K$ and a Weierstrass model*

$$E : Y^2 = 4X^3 - g_2 X - g_3 \qquad g_2, g_3 \in \mathbb{F}$$

Then, for $w, w' \in \Gamma \otimes \frac{1}{n}\mathbb{Z}$, the Laurent expansion of the function $\Theta_{(w,w')}(z, z')$ has coefficients in $\mathbb{F}(E[4n^2])$.

In particular, this corollary and theorem 4.3.1 show that the Eisenstein-Kronecker numbers

$$\frac{e^*_{a,b}(z_0, z_0'; \Gamma)}{a! A(\Gamma)^a} \qquad \text{are algebraic.} \tag{5.10}$$

Now let χ be a Hecke character of K, with conductor $\mathfrak{f}$ and ∞-type $(a, -b)$ as in §3.2.11. Let $\Omega \in \mathbb{C}$ be a a complex number such that $\Gamma = \Omega\mathfrak{f}$. By (3.7) one has

$$L_{\mathfrak{f}}(0, \chi) = \frac{1}{\omega_{\mathfrak{f}}} \sum_{\mathfrak{a} \in \mathfrak{Cl}_{\mathfrak{f}}} \chi(\mathfrak{a}) e^*_{a,b}\left(\alpha_{\mathfrak{a}}, 0; \mathfrak{f}\mathfrak{a}^{-1}\right)$$

where, $\Omega\mathfrak{f}\mathfrak{a}^{-1}$ is also a period lattice for some Weierstrass model E', defined over a number field K'. From (5.10) one deduces that

$$\frac{e^*_{a,b}\left(\alpha_{\mathfrak{a}}, 0; \Omega\mathfrak{f}\mathfrak{a}^{-1}\right)}{A(\Omega\mathfrak{f}\mathfrak{a}^{-1})^a} = \frac{\overline{\Omega}^a}{\Omega^b} \frac{e^*_{a,b}\left(\alpha_{\mathfrak{a}}, 0; \mathfrak{f}\mathfrak{a}^{-1}\right)}{A(\Omega\mathfrak{f}\mathfrak{a}^{-1})^a} \qquad \text{are algebraic.}$$

On the other hand

$$A(\Omega\mathfrak{f}\mathfrak{a}^{-1})^a = N(\mathfrak{f}\mathfrak{a}^{-1})|\Omega| A(\mathcal{O}_K) = N(\mathfrak{f}\mathfrak{a}^{-1})|\Omega| \frac{\sqrt{d_K}}{2\pi}$$

Hence, for $a, b \geq 0$ with $b - a \geq 0$ the numbers

$$\left(\frac{2\pi}{\sqrt{d_K}}\right)^a \frac{e^*_{a,b}\left(\alpha_{\mathfrak{a}}, 0; \mathfrak{f}\mathfrak{a}^{-1}\right)}{\Omega^{a+b}} \qquad \text{are algebraic.}$$

This provides another proof of Damerell's theorem (theorem 5.1.1).

6
Conclusion and further readings

The topic of L-functions is one of the most interesting and fascinating topics in modern mathematics. Although they have countless generalisations, the nature of their special values are junctions of multiple mathematical areas that hide precious treasure underneath.

6.1. Summary

In our case, the study of the Eisenstein-Kronecker numbers, tied to Hecke L-functions on imaginary quadratic field led us —thanks to the key observation made by Bannai and Kobayashi in [BK+10]— to explore and use a new approach of investigating these special values through Mumford's theory of algebraic theta functions. This turns out to have particular applications as we will see. We summarise the work done here as follow:

- Study of the Eisenstein-Kronecker-Lerch series through the very rich work done by Eisenstein and Kronecker (and nicely wrapped up by Wale): We show that this series, in a number theoretic setting, admits an analytic continuation and satisfies a functional equation.
- Relating the Eisenstein-Kronecker-Lerch series $K_1(z, z', 1)$ to the Kronecker theta function $\Theta(z, z')$, which turns out to be a reduced theta function the unique reduced theta function on $E \times E^\vee$.
- Study Theta functions ϑ_s as meromorphic sections s of a line bundle on an abelian variety, and use Mumford's theory to construct an algebraic theta-translation operator, that preserves algebraicity.
- Applying this theory to the case of the Poincaré bundle of an elliptic curve with CM: We show that the Laurent series of the algebraic translation of the Kronecker theta function $\overset{\mathcal{M}}{\Theta}_{(w,w')}(z, z')$ by rational points $w, w' \in \Gamma \otimes \mathbb{Q}$ has algebraic coefficients, which shows the algebraicity of the Eisenstein-Kronecker numbers $e^*_{a,b}(w, w')$.

6.2. Difficulties Encountered

The difficulties encountered in this process were not insurmountable. The first technicalities about the singularities of the series $K^*_{a,b}$ were not too hard to work around in order to prove the main result. The main difficulty was understanding and adapting Mumford's algebraic theory to a general abelian variety $A(k)$ on a general ground field: I would highlight the not so obvious task of making sure the translation of a point —not in A but from its universal cover— is canonically compatible with our previous constructions. Although the translation initially defined also preserves algebraicity (as shows eq. (5.9)), using algebraic theta functions had much much more interesting things to offer.

6.3. Outlook: p-adic interpolation of Hecke L-functions

Going back to the Birch and Swinneton-Dyer's conjecture for an elliptic curve $E(\mathbb{Q})$:

Conjecture 6.3.1 (Birch and Swinneton-Dyer). *Let E be an elliptic curve over $\mathbb{Q}$, then the Taylor expansion of its L-function is given by:*

$$L(E(\mathbb{Q}),s) = \frac{1}{\#(E(\mathbb{Q})_{tors})^2} \cdot \begin{pmatrix} \textit{something conjectured} \\ \textit{to be finite} \end{pmatrix} \cdot \prod_p c_p (s-1)^r + \textit{ higher terms.}$$

where $r = \operatorname{rank} E(\mathbb{Q})$.

This refinement is due to Wiles (see [CWC+06]). Since the sum does not converge for $s = 1$, it is conjectured however (conjecture 3.3.13) that $L(s,1)$ has analytic continuation to all $\mathbb{C}$. Based on numerical evidence, Birch and Swinnerton-Dyer suggested the approximation

$$\prod_{p \le x} \frac{E(\mathbb{F}_p)}{p} \approx C \cdot \log(x)^r \quad \text{ when } x \longrightarrow \infty$$

This leads to consider p-adic functions that interpolate the values of $L(E(\mathbb{Q}),s)$.

The idea behind is the following: when one studies the special values of L-functions, and (using analytic continuation and functional equation) obtains —up to a controllable factor— algebraic numbers, this suggests that in order to understand these values, it should be enough to understand them locally at each prime.

Now, given a sequence of p-adic numbers $(x_n)_{n\in\mathbb{Z}}$, the concept of p-adic interpolation is the following: If one considers x_n as function $x_n : \mathbb{Z} \longrightarrow \mathbb{Q}_p$, the goal would be to extend it (if possible) to a continuous p-adic function $x_{n,p} : \mathbb{Z}_p \longrightarrow \mathbb{Q}_p$. Another way to look at it, is to construct a p-adic measure i.e. a continuous linear map

$$\mu : \mathscr{C}(\mathbb{Z}_p, \mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$$

whose moments interpolate the values of $L(E(\mathbb{Q}),s)$.

The following approach; using the algebraic Kronecker theta functions through Mumford theory; allows to show the p-integrality of its Laurent series, when $p \ge 5$ is a prime that splits into $\mathfrak{p}\overline{\mathfrak{p}}$ in $\mathcal{O}_K$ ([BK+10], Corollary 2.17). Bannai and Kobayashi use this theorem to construct a p-adic measure $\mu_{w,w'}$ on $\mathbb{Z}_p \times \mathbb{Z}_p$; for $w, w' \in \Gamma \otimes \mathbb{Q}$; that p-adically interpolates the Eisenstein-Kronecker numbers. Their measure $\mu_{w,w'}$ may also be used to construct the two-variable p-adic L-function for algebraic Hecke characters on imaginary quadratic fields, known from Manin-Vishik and from Katz. The approach using the generating function gives more details however, on the p-adic properties of $e^*_{a,b}$ even when the prime p (called *supersingular* case) does not split and remains prime in K. (see §4 [BK+10]).

A

Appendix A : Analytical tools

A.1. Convergence results and Fourier series on $\mathbb{T}$

Let $\Gamma = u\mathbb{Z} \oplus v\mathbb{Z} \subset \mathbb{C}$ be a lattice with generators $u, v \in \mathbb{C}$ such that $\Im(\tau) > 0$, $\tau = \frac{v}{u}$. Let

$$A = \frac{(\text{Area of } \mathbb{C}/\Gamma)}{\pi} = \frac{1}{\pi}\Im(\tau) = \frac{v\overline{u} - u\overline{v}}{2\pi i} \quad (> 0).$$

Lemma A.1.1 (Convergence Lemma)**.** *For $k \geq 3$ the series $\sum\limits_{\substack{\gamma\in\Gamma\\ \gamma\neq 0}} \frac{1}{\gamma^k}$ is absolutely convergent.*

Proof. We will show first that, the number of elements $w \in \Gamma$ such that $|\gamma|$ is between two consecutive integers n and $n+1$ is a $O(n)$ i.e that there is a constant K_Γ (that depends only on the lattice Γ) such that there are at most nK_Γ points with $n < |\gamma| < n+1$. Indeed, for x, y in Γ there exists a $\delta > 0$ such that $|x - y| > \delta$. Thus for all $0 < \epsilon < 1$ and for all $\gamma \in \Gamma$

$$B(\gamma, \epsilon\delta) \cap \Gamma = \{\gamma\}$$

Let $R_n := \{z \in \mathbb{C} \mid n < |z| < n+1\}$ be some annulus or "ring". Then for every $\gamma \in \Gamma \cap R_n$

$$\bigcup_{\gamma\in\Gamma\cap R_n} B(\gamma, \epsilon\delta) \subset R'_n := \{z \in \mathbb{C} \mid n - \epsilon\delta < |z| < n + 1 + \epsilon\delta\} \quad \text{(the union is even disjoint}$$

But, the air of R'_n is given by

$$A(R'_n) = \pi(n + 1 + \epsilon\delta)^2 - \pi(n - \epsilon\delta)^2 = \pi(2n + 1)(1 + 2\epsilon\delta)$$

Hence

$$\#\{\gamma \in \Gamma \cap R\} \leq \left\lfloor \frac{A(R'_n)}{A(B(\gamma, \epsilon\delta))} \right\rfloor = \left\lfloor \frac{1 + 2\epsilon\delta}{\epsilon^2\delta^2}(2n + 1) \right\rfloor = O(n).$$

Observe that δ does not depend on n, and if $\epsilon \to 1$, then there is constant K_Γ such that

$$\#\{\gamma \in \Gamma \cap R\} \leq \left\lfloor \frac{1 + 2\delta}{\delta^2}(2n + 1) \right\rfloor = nK_\Gamma$$

Now, we can write the sum

$$\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq 0}} \frac{1}{\gamma^k} = \sum_{n\in\mathbb{N}} \sum_{n<|\gamma|<n+1} \frac{1}{\gamma^k}$$

and it suffices to show the absolute convergence of the series

$$\sum_{n<|\gamma|<n+1} \frac{1}{\gamma^k}$$

Clearly,

$$\sum_{n<|\gamma|<n+1} \frac{1}{|\gamma^k|} \leq \sum_{n<|\gamma|<n+1} \frac{1}{n^k} < nK_\Gamma \frac{1}{\gamma^k} = \frac{K_\Gamma}{n^{k-1}}$$

and we get

$$\sum_{n\in\mathbb{N}} \sum_{n<|\gamma|<n+1} \frac{1}{|\gamma|^k} \leq \sum_{n\in\mathbb{N}} \frac{k_\Gamma}{n^{k-1}} \quad \text{which converges for all } k \geq 3.$$

□

We defined the pairing for complex numbers w, z by

$$\langle z, w \rangle_\Gamma := \exp\left(\frac{\overline{w}z - w\overline{z}}{A}\right).$$

We identify the complex torus $\mathbb{T} := {}^{\mathbb{C}}\!/_{\Gamma}$ with its dual $\mathbb{T}^\vee$ through the isomorphism

$$\begin{aligned} \mathbb{T} &\longrightarrow \operatorname{Hom}(\mathbb{T}, \mathbb{C}^\times) \\ z &\longmapsto (w \mapsto \langle w, z \rangle_\Gamma) \end{aligned}$$

Hence, for a sufficiently well behaved function f (rapidly decreasing, smooth) the *Fourier transform* of $f(z)$ is given by

$$\widehat{f}(w) = \int_{\mathbb{T}} f(z)\overline{\langle w, z \rangle}\, dz = \int_{\mathbb{T}} f(z) \langle z, w \rangle\, dz$$

Proposition A.1.2 (Poisson summation for Γ)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a smooth and fast decaying function, then*

$$\sum_{\gamma\in\Gamma} f(\gamma) = \frac{1}{|\Gamma|} \sum_{\gamma^*\in\Gamma^\vee} \widehat{f}(\gamma^*)$$

Proof. Pose $g : x \longmapsto \sum_{\gamma\in\Gamma} f(x+\gamma)$, g is clearly smooth and Γ-periodic, hence g admits the following Fourier series expansion

$$g(x) = \sum_{\gamma^*\in\Gamma^\vee} a_\gamma e^{2\pi i(x\cdot\gamma)}$$

where

$$\begin{aligned} a_\gamma &= \frac{1}{|\Gamma|} \int_{{}^{\mathbb{R}^n}\!/_{\Gamma}} g(x) e^{-2\pi i(x\cdot\gamma)}\, dx = \frac{1}{|\Gamma|} \sum_{\gamma^*\in\Gamma^\vee} \int_{{}^{\mathbb{R}^n}\!/_{\Gamma}} f(x) e^{-2\pi i(x\cdot\gamma)}\, dx \\ &= \frac{1}{|\Gamma|} \sum_{\gamma^*\in\Gamma^\vee} \int_{\mathbb{R}^n} f(x) e^{-2\pi i(x\cdot\gamma)}\, dx = \frac{1}{|\Gamma|} \widehat{f}(\gamma). \end{aligned}$$

Hence

$$\sum_{\gamma\in\Gamma} f(\gamma) = g(0) = \sum_{\gamma^*\in\Gamma^\vee} a_{\gamma^*} = \frac{1}{|\Gamma|}\sum_{\gamma^*\in\Gamma^\vee} \widehat{f}(\gamma^*).$$

□

Definition A.1.3 (Jacobi theta function)**.** The **Jacobi theta function** is the (holomorphic) function defined on the right half plane $\{z \in \mathbb{C} \mid \Re(z) > 0\}$ by

$$\theta(s;\Gamma) = \sum_{\gamma\in\Gamma} e^{-\pi s(\gamma\cdot\gamma)}$$

Now in the context of §2.3.1 we define the *modified* Jacobi theta function to be

$$\theta_{a,t}(z,z') = \sum_{\gamma\in\Gamma} \exp\left(-t|z+\gamma|^2\right)(\overline{z}+\overline{\gamma})^a\langle\, \gamma, z'\,\rangle$$

Lemma A.1.4. *The above function satisfies the following functional equation*

$$\theta_{a,t}(z_0,z_0') = \frac{\langle\, z_0', z_0\,\rangle}{(At)^{a+1}}\theta_{a,A^{-2}t^{-1}}(z_0',z_0)$$

Proof. For simplicity, we pose for fixed $t \in \mathbb{R}_{>0}$ and z_0, w_0 in $\mathbb{C}$:

$$f(z) := \exp\left(-\frac{|z|^2}{2}\right) \qquad h(z,t) := f\left(z\sqrt{tA}\right)$$

$$h^+(z,z_0,t) := h(z+z_0,t) \qquad k(z,z_0,w_0,t) := h^+(z,z_0,t)\langle\, z, w_0\,\rangle$$

First, observe that $\widehat{f} = f$: This is a known fact about the so called *Gaussian function* f and simply reduces to the one variable case $\widehat{f}(w) = \prod_1^n I(w)$ where

$$I(w) = \exp\left(-\pi\gamma^2\right)\int_{\mathbb{R}} \exp\left(-\pi(z+iw)^2\right)dz = \exp\left(-\pi w^2\right)$$

On the other hand

$$\widehat{h}(w,t) = \int_{\mathbb{T}} f\left(z\sqrt{tA}\right)\langle\, z, w\,\rangle\, dz = \frac{1}{tA}\int_{\mathbb{T}} f(x)\langle\, x, \frac{w}{\sqrt{tA}}\,\rangle\, dx \qquad \text{by performing the change of variable } x = z\sqrt{tA}$$

$$= \frac{1}{tA}\widehat{f}\left(\frac{w}{\sqrt{tA}}\right) = \frac{1}{tA}\exp\left(-\frac{|w|^2}{tA^2}\right)$$

Thus

$$\widehat{k}(w,z_0,w_0,t) = \int_{\mathbb{T}} k(w,z_0,w_0,t)\langle\, z,w\,\rangle\, dz = \int_{\mathbb{T}} h^+(z,z_0,t)\langle\, z, w+w_0\,\rangle\, dz = \widehat{h^+}(w+w_0,t)$$

Where

$$\widehat{h^+}(w,z_0,t) = \int_{\mathbb{T}} h(z+z_0,t)\langle\, z,w\,\rangle\, dz = \int_{\mathbb{T}} h(x,t)\langle\, x-z_0, w\,\rangle\, dx \qquad \text{by performing the change of variable } x = z+z_0$$

$$= \langle\, w, z_0\,\rangle\widehat{h}(w,t) = \frac{\langle\, w, z_0\,\rangle}{tA}\exp\left(-\frac{|w|^2}{tA^2}\right)$$

Thus one finally gets

$$\widehat{k}(w, z_0, w_0, t) = \frac{\langle\, w + w_0, z_0 \,\rangle}{tA} \exp\left(-\frac{|w + w_0|^2}{tA^2}\right) = \frac{\langle\, w_0, z_0 \,\rangle}{tA} k(w, w_0, z_0, A^{-2}t)$$

Now, applying Poisson summation we get

$$\begin{aligned}\theta_{0,t}(z_0, z_0') &= \sum_{\gamma\in\Gamma} k(\gamma, z_0, w_0, t) = \frac{1}{tA}\sum_{\gamma\in\Gamma}\langle\, w_0, z_0 \,\rangle k(\gamma, w_0, z_0, A^{-2}t) \\ &= \frac{1}{tA}\theta_{0,A^{-2}t^{-1}}(z_0', z_0)\end{aligned}$$

Hence, the result follows by induction on a, and taking the derivatives $\frac{d}{dz}$. □

A.2. Weierstrass's Elliptic functions

One of the most famous elliptic functions are Weirstrass's $\wp$, σ and ζ functions, defined by

$$\begin{aligned}\wp(z, \Gamma) &:= \frac{1}{z^2} + \sum_{\gamma\in\Gamma\smallsetminus\{0\}}\left(\frac{1}{(z+\gamma)^2} - \frac{1}{\gamma^2}\right) \\ \sigma(z, \Gamma) &:= z\prod_{\gamma\in\Gamma\smallsetminus\{0\}}\left(1 - \frac{z}{\gamma}\right)\exp\left(\frac{z}{\gamma} + \frac{z^2}{2\gamma^2}\right) \\ \zeta(z, \Gamma) &:= \frac{1}{z} + \sum_{\gamma\in\Gamma\smallsetminus\{0\}}\left(\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2}\right)\end{aligned}$$

where, the ζ function satisfies for $z \in \mathbb{C}\smallsetminus\Gamma$

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} \quad \text{and} \quad \zeta'(z) = -\wp(z)$$

Observe that

$$\begin{aligned}\frac{\frac{d}{dz}\left(\frac{\sigma(z+\gamma)}{\sigma(z)}\right)}{\frac{\sigma(z+\gamma)}{\sigma(z)}} &= \frac{d}{dz}\left(\ln\frac{\sigma(z+\gamma)}{\sigma(z)}\right) = \eta(\gamma) \\ \Rightarrow \quad & \frac{\sigma(z+\gamma)}{\sigma(z)}e^{-\eta(\gamma)z} = e^{\delta}\end{aligned}$$

where δ is some constant and η is the quasi-period (or Weirstrass's eta function) defined by

$$\begin{aligned}\eta : \Gamma &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto \eta(\gamma) := \zeta(z+\gamma) - \zeta(z).\end{aligned}$$

The function η is independent* from z and clearly $\mathbb{Z}$-linear, thus it is entirely determined by the periods (ω_1, ω_2). Moreover, it satisfies the *Legendre formula*

$$\eta(\omega_2)\omega_1 + \eta(\omega_1)\omega_2 = 2\pi i$$

To see that, choose a parallelogram with vertices ABCD as in the figure A.1. Then by Cauchy's theorem one has

$$\int_{ABCDA} \zeta(z)\, dz = 2\pi i$$

On the other hand, one sees that

$$\begin{aligned}\int_{CD} \zeta(z)\, dz &= \int_{BA} \zeta(z+\omega_2)\, dz \\ &= \int_{BA} \zeta(z) + \eta(\omega_2)\, dz \\ &= \int_{BA} \zeta(z)\, dz - \eta(\omega_2)\omega_1 .\end{aligned}$$

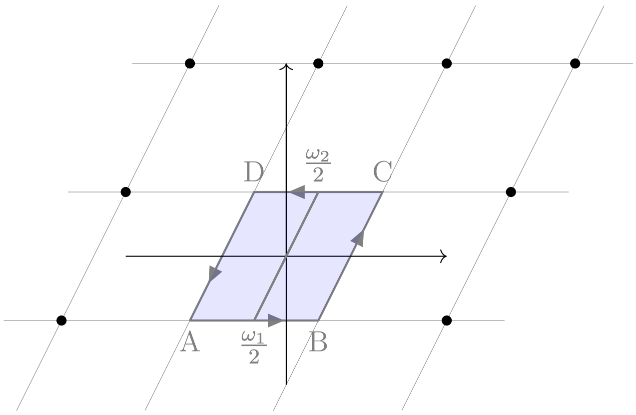

Figure A.1.: Parallelogram ABCD

This implies that

$$\int_{AB} \zeta(z)\, dz + \int_{CD} \zeta(z)\, dz = -\eta(\omega_2)\omega_1$$
$$\int_{BC} \zeta(z)\, dz + \int_{DA} \zeta(z)\, dz = \eta(\omega_1)\omega_2$$

Finally, we present the Laurent expansion around $z = 0$ of the Weierstrass's σ- function, given in [WM66](§10.5 p.391) or [MA70](18.5.6 p.635-636) by

$$\sigma(z) = \sum_{m,n\geq 0} a_{m,n} \left(\frac{g_2}{2}\right)^m (2g_3)^n \frac{z^{4m+6n+1}}{(4m+6n+1)!} \tag{A.1}$$

where

$$\begin{cases} a_{0,0} = 1 & \\ a_{m,n} = 0 & \text{if } m < 0 \text{ or } n < 0 \\ \begin{aligned} a_{m,n} &= 3(m+1)a_{m+1,n+1} + \tfrac{16}{3}(n+1)a_{m-2,n+1} \\ &\quad -\tfrac{2}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n} \end{aligned} & \text{if } m,n > 0 \end{cases}$$

*Indeed, one easily checks that for $\gamma \in \Gamma$: $\frac{d}{dz}(\zeta(z+\gamma) - \zeta(z)) = -\wp(z+\gamma) + \wp(z) = 0$

B

Appendix B : Review of Algebraic Geometry

B.1. Class field theory

We provide a review of the concepts which have been used (implicitly or explicitly) during the development of chapter 3 and chapter 4. Most of the proofs can be found in the very complete and classic [Mil08b] or any other textbook on algebraic number theory. Let K be a number field, $K_\mathfrak{p}$ its completion at each place $v_\mathfrak{p}$. Moreover, let $\mathcal{O}_\mathfrak{p}$ be the ring of integers of $K_\mathfrak{p}$ for all non-archimedean places, and let $\mathcal{O}_v = K_v$ otherwise.

Definition B.1.1. Let L/K be a finite Galois extension of number fields with Galois group G, $\mathfrak{p}$ an unramified prime of K in L and $\mathfrak{P}$ the prime of L lying over $\mathfrak{p}$. Let $l = {}^{\mathcal{O}_L}\!/_{\mathfrak{P}}$, $k = {}^{\mathcal{O}_K}\!/_{\mathfrak{p}}$ be the residue fields of $L_\mathfrak{P}$ and $K_\mathfrak{p}$ respectively. Then one has an isomorphism

$$G(\mathfrak{P}) \cong \mathrm{Gal}(L_\mathfrak{P}/K_\mathfrak{p}) \cong \mathrm{Gal}(l/k)$$

where $G(\mathfrak{P}) = \{\sigma \in \mathrm{Gal}(L/K) \,/\, \sigma\mathfrak{P} = \mathfrak{P}\}$ is the decomposition group of $\mathfrak{P}$ and $\mathrm{Gal}(l/k)$ is generated by the Frobenius automorphisms $\phi_{L/K}$ defined by the formula

$$\phi_{L/K}(a) = a^q \quad \text{for all } a \in L.$$

The image of such an automorphism of $\mathrm{Gal}(l/k)$ in $\mathrm{Gal}(L/K)$, that generates $G(\mathfrak{P})$, is called the **Frobenius element** of $\mathrm{Gal}(L/K)$ at $\mathfrak{P}$ and denoted $(\mathfrak{P}, L/K)$.

For any $\sigma \in \mathrm{Gal}(L/K)$ and $\alpha \in \mathcal{O}_L$ one has

$$\sigma(\mathfrak{P}, L/K)\sigma^{-1} \cdot \alpha \equiv \sigma(\sigma^{-1} \cdot \alpha)q \equiv \alpha^q \mod \sigma\mathfrak{P} \quad \Rightarrow \quad \sigma(\mathfrak{P}, L/K)\sigma^{-1} = (\sigma\mathfrak{P}, L/K)$$

where $q = \#k$. Moreover, if L/K is abelian, then the Frobenius elements $(\mathfrak{P}, L/K)$ for all $\mathfrak{P}$ over $\mathfrak{p}$ are equal and one simply writes $(\mathfrak{p}, L/K)$.

Definition B.1.2 (Artin reciprocity map)**.** Let L/K be a finite abelian extension, and let S be a set of primes of K containing all the primes that ramify in L. Let I_K^S be the subgroup of the ideal group I_K generated by $\mathfrak{p}$ in $K \smallsetminus S$. Define the **Artin reciprocity map** to be the surjective homomorphism

$$\begin{aligned}(\,\cdot\,, L/K) :\ I_K^S &\longrightarrow \mathrm{Gal}(L/K)\\ \prod_{i=1}^{t} \mathfrak{p}_i^{n_i} &\longmapsto \prod_{i=1}^{t} (\mathfrak{p}_i, L/K)^{n_i}\end{aligned}$$

The Artin map factors through

$$\begin{tikzcd} I_K^S \arrow[r, "N_{L/K}"] \arrow[dr, "(\cdot, L/K)"'] & N_{L/K}(I_L^S) \arrow[d] \\ & \mathrm{Gal}(L/K) \end{tikzcd}$$

where I_L^S is generated by the primes $\mathfrak{P}$ over $\mathfrak{p} \in S$. (this is due to the fact that $N_{L/K}(\mathfrak{P}) = \mathfrak{p}^{f(\mathfrak{P}/\mathfrak{p})}$ for $\mathfrak{P}$ over $\mathfrak{p}$ in K.)

In general, when K is not totally imaginary, one has a notion of a **modulus** $\mathfrak{m}$ (when K is totally imaginary, it is just an integral ideal)

Definition B.1.3 (Modulus function)**.** A modulus of K is a function

$$\mathfrak{m} : \{ \text{ places of } K \} \longrightarrow \mathbb{N}$$

satsifying:

$$\begin{cases} \mathfrak{m}(\mathfrak{p}) = 0 & \text{for almost all} \\ \mathfrak{m}(\mathfrak{p}) \in \{0,1\} & \text{if } v_{\mathfrak{p}} \text{ is real} \\ \mathfrak{m}(\mathfrak{p}) = 1\} & \text{if } v_{\mathfrak{p}} \text{ is complex} \end{cases}$$

One usually writes the modulus as a formal product

$$\mathfrak{m} = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathfrak{m}(\mathfrak{p})}.$$

Define $S(\mathfrak{m}) := \{\mathfrak{p} : \mathfrak{m}(\mathfrak{p}) > 0\}$ and let $P_{\mathfrak{m}}$ be the group of principal fractional ideals (α) such that α is positive under all real embeddings of K, and $\alpha - 1 \in \mathfrak{p}^{\mathfrak{m}(\mathfrak{p})}$ for all finite $v_{\mathfrak{p}}$. The quotient

$$C_{\mathfrak{m}}(K) := I_K^{S(\mathfrak{m})} \big/ P_{\mathfrak{m}}$$

is called the **ray class group** of K, modulo $\mathfrak{m}$.

Theorem B.1.4 (Artin reciprocity)**.** *Let L/K be a finite abelian extension. Then there exists a modulus $\mathfrak{m}$ such that $S(\mathfrak{m})$ contains all primes of K that ramify in L and $P_{\mathfrak{m}} \subseteq \ker(\cdot, L/K)$. The Artin map factors to an isomorphism*

$$I_K^{S(\mathfrak{m})} \big/ P_{\mathfrak{m}} N_{L/K}(I_L^{S(\mathfrak{m})}) \cong \mathrm{Gal}(L/K)$$

In particular, for a congruence subgroup H with

$$P_{\mathfrak{m}} \subseteq H \subseteq I_K^{S(\mathfrak{m})}$$

Takagi showed ([Tak22] which goes by the name of the *existence theorem*) that there exists a finite abelian extension L/K such that

$$H = P_{\mathfrak{m}} N_{L/K}(I_L^{S(\mathfrak{m})})$$

By Artin reciprocity, one has then an isomorphism

$$I_K^{S(\mathfrak{m})} \Big/ H \cong \mathrm{Gal}(L/K).$$

In this case, L is called the **class field** of H.

Remark B.1.5. (i) An important example occurs when $H = C_{\mathfrak{m}}$, and the class field K_m of $P_{\mathfrak{m}}$ is then called the **ray class field modulo $\mathfrak{m}$**. In this case, the Artin map induces an isomorphism

$$C_{\mathfrak{m}} \cong \mathrm{Gal}(K_{\mathfrak{m}}/K)$$

between the ray class group and $\mathrm{Gal}(K_{\mathfrak{m}}/K)$.

(ii) When $\mathfrak{m} = 1$, the ray class field of K is called the **Hilbert class field** K_H of K, and is therefore, the maximal unramified abelian extension of K. The Artin reciprocity gives in this case, an isomorphism

$$\mathrm{Cl}(K) \cong Gal(K_H/K)$$

between the ideal class group of K and $\mathrm{Gal}(K_H/K)$.

The idèlic formulation of class field theory, which provides a perspective of class field theory in terms of idèles rather than in terms of ideals, is based on the Artin reciprocity map described in the following:

Definition B.1.6. Let L/K be a finite field extension. The norm map is given by

$$\begin{aligned} N_{L/K} : \mathbb{J}_L &\longrightarrow \mathbb{J}_K \\ (s_{\mathfrak{P}})_{\mathfrak{P}} &\longmapsto (\prod_{\mathfrak{P}|\mathfrak{p}} N_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} s_{\mathfrak{P}})_{\mathfrak{p}} \end{aligned}$$

Theorem B.1.7 (Artin reciprocity - idèles)**.** *Let K^{ab} be the maximal abelian extension of K. There is a unique continuous homomorphism*

$$\begin{aligned} [\cdot, K] : \mathbb{J}_K &\longrightarrow \mathrm{Gal}(Kab/K) \\ s &\longmapsto [s, K] \end{aligned}$$

such that for any finite abelian extension L/K and idèle $s \in \mathbb{J}_L$ where (s) is not divisible by any prime that ramify in L:

$$[s, K]_{|L} = ((s), L/K)$$

Moreover,

(i) $[\cdot, K]$ *is surjective and* $K^\times \subseteq \ker[\cdot, K]$.

(ii) If L/K *is a finite abelian extension, then*

$$[s, L]_{|K^{ab}} = [N_{L/K}(s), K] \quad \textit{for all } s \in \mathbb{J}_L.$$

(iii) If $\mathfrak{p}$ *is a prime of* K *and* L/K *is an abelian extension that does not ramify at* $\mathfrak{p}$*, then*

$$[\pi, K]_{|L} = (\mathfrak{p}, L/K),$$

where $\pi = (1, \ldots, \underset{\substack{\uparrow \\ \mathfrak{p}\textit{th component}}}{\pi_{\mathfrak{p}}}, \ldots, 1)$ *is an idèle of* K *and* $\pi_{\mathfrak{p}}$ *a uniformizer of* $\mathcal{O}_{\mathfrak{p}}$.

If $K_{\mathfrak{m}}$ is the ray class field of K modulo some modulus $\mathfrak{m}$ and

$$U_{\mathfrak{m}} := \{s \in \mathbb{J}_K \,/\, s_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^\times, s_{\mathfrak{p}} - 1 \in \mathfrak{p}^{\mathfrak{m}(\mathfrak{p})} \text{ for all } v_{\mathfrak{p}} \nmid \infty, s_{\mathfrak{p}} > 0 \text{ for all real } v_{\mathfrak{p}}\},$$

which is an open subgroup of $\mathbb{J}_K$, then the Artin map factors to an isomorphism

$$[\cdot, K] : \mathbb{J}_K / K^\times U_{\mathfrak{m}} \cong \mathrm{Gal}(K_{\mathfrak{m}}/K).$$

B.2. Algebraic abelian varieties

Through all this section, k will denote an algebraically closed field.

Definition B.2.1. An **algebraic variety** $A(k)$ over a field k is an integral and separated scheme over k which is of finite type.
For a scheme S, a **Group scheme** or an **algebraic group** is a group object in the category of schemes over k, i.e. an S-scheme $\mathcal{G}$ with morphisms

$$m : \mathcal{G} \times_S \mathcal{G} \longrightarrow \mathcal{G} \quad , \quad e : S \longrightarrow \mathcal{G} \quad \text{and} \quad inv : \mathcal{G} \longrightarrow \mathcal{G}$$

satisfying the group axioms.

A **homomorphism of group schemes** $f : \mathcal{G} \longrightarrow \mathcal{G}'$ is a morphism of S-schemes such that

$$f \circ e = e', \quad m' \circ f^2 = f \circ m \quad \text{and} \quad inv' \circ f = f \circ inv$$

One defines the kernel of such a homomorphism, to be the subgroup scheme satisfying the fibre product condition:

$$\begin{tikzcd} \ker f \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "f"] \\ S \arrow[r, "e'"] & \mathcal{G}' \end{tikzcd}$$

A noticeable example of a group scheme is the multiplicative group $\mathbb{G}_m$ over the base scheme $\mathbb{Z}$, given by $\mathbb{G}_m \cong \operatorname{Spec}\mathbb{Z}[T, T^{-1}]$ and

$$\begin{array}{rcl} e : \mathbb{Z}[T,T^{-1}] & \longrightarrow & \mathbb{Z}, \\ T & \longmapsto & 1 \end{array} \qquad \begin{array}{rcl} inv : \mathbb{Z}[T,T^{-1}] & \longrightarrow & \mathbb{Z}, \\ T & \longmapsto & T^{-1} \end{array} \qquad \begin{array}{rcl} m : \mathbb{Z}[T,T^{-1}] & \longrightarrow & \mathbb{Z}[T,T',T^{-1},T^{-1}] \\ T & \longmapsto & TT' \end{array}$$

Definition B.2.2 (Abelian variety)**.** An (algebraic) **abelian variety** $X(k)$ is a group scheme which is a complete variety over k.

A **homomorphism of abelian varieties** $f : X(k) \longrightarrow Y(k)$ is a homomorphism of the underlying group schemes.

one has an important result:

Lemma B.2.3 (Rigidity lemma)**.** *Let $f : X(k) \times Y(k) \longrightarrow Z(k)$ be a morphism of varieties over k. Suppose X is proper and that $f(X(k) \times \{y_0\}) = f(\{x_0\} \times Y(k)) = \{z_0\}$ for some $x_0 \in X(K), y_0 \in Y(k)$ and $z_0 \in Z(k)$. Then*

$$f(X \times Y) = \{z_0\}$$

Proof. Let $U(Z)$ be an affine neighbourhood of z_0. Since X is proper, $p_2 : X \times Y \longrightarrow Y$ is closed, thus $A = p_2\left(f^{-1}(Z \smallsetminus U(Z))\right) \subset Y$ is closed. By assumption, $y_0 \in Y \smallsetminus A$ which makes it dense.

Let $y \in Y \smallsetminus A$. Observe first that $f(A \times \{y\}) \subset U(Z)$. As X is proper and $U(Z)$ affine, $f(A \times \{y\})$ is reduced to a point and

$$f(\{x_0\} \times \{y\}) \in f(A \times \{y\}) \cap f(\{x_0\} \times Y).$$

Hence, by assumption, $\{z_0\} = f(A \times \{y\})$ for all $y \in Y \smallsetminus A$. Thus f is constant on $X \times (Y \smallsetminus A)$ and by density the result follows. □

In particular, one has the following:

Proposition B.2.4 (Morphisms of abelian varieties)**.** *Let $f : X(k) \longrightarrow Y(k)$ be a morphism of abelian varieties, then f is the composition of a homomorphism and a translation.*

Proof. Let $g : x \longmapsto f(x) - f(0)$ and consider

$$\begin{aligned} h : X(k) \times X(k) &\longrightarrow Y \\ (x, x') &\longmapsto g(x) + g(x') - g(x + x') \end{aligned}$$

Clearly, $h\left(X(k) \times \{e_X\}\right) = f\left(\{e_X\} \times X(k)\right) = \{e_Y\}$. Thus, by the rigidity lemma, one gets

$$g(x) + g(x') = g(x + x')$$

□

Remark B.2.5. Note that, by applying the same reasoning, the rigidity lemma (applied to the map $inv : X(k) \longrightarrow X(k)$) implies that the group law of an abelian variety is always commutative. This somehow justifies the choice of the nomenclature.

For any point $a \in X(k)$ we define the translation morphism

$$\begin{aligned} t_a : X(k) &\longrightarrow X(k) \\ x &\longmapsto x + a \end{aligned}$$

and for all integer n the endomorphism

$$\begin{aligned} [n] : X(k) &\longrightarrow X(k) \\ x &\longmapsto nx \end{aligned}$$

Theorem B.2.6 (Theorem of the square)**.** *For a line bundle $\mathcal{L}$ on an abelian variety $X(k)$ and for all points $x, x' \in X(k)$ one has*

$$t^*_{x+x'}\mathcal{L} \cong t^*_x \otimes t^*_{x'_0}$$

Proof. For a proof, see [DM70]. □

Mumford gives an algebraic proof of this theorem. In his proofs, he makes use of a very useful and technical Lemma (Corollary 6 p.54, [DM70]) known as the Seesaw principle:

Lemma B.2.7 (Seesaw principle)**.** *For a complete variety $X(k)$ and a line bundle $\mathcal{L}$ on $X \times T$ where T is any variety. The set*

$$T_1 = \{t \in T \,|\, \mathcal{L} \textit{ is trivial on } X \times T\}$$

is closed in T. Moreover

$$\mathcal{L}_{|X \times T_1} \cong p_2^* \mathcal{M}$$

for some line bundle $\mathcal{M}$ on T_1 (with the reduced scheme structure).

In particular, one has the following corollaries:

Corollary B.2.8. *For any line bundle $\mathcal{L}$ on $X(k)$ and any integer n one has*

$$[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{n^2+n}{2}} \otimes inv^*\mathcal{L}^{\otimes \frac{n^2-n}{2}}.$$

Corollary B.2.9. *For any line bundle $\mathcal{L}$ on an abelian variety $X(k)$ the map*

$$\begin{aligned} \phi_{\mathcal{L}} : X(k) &\longrightarrow \mathrm{Pic}(X) \\ x &\longmapsto t^*_x \otimes \mathcal{L}^{-1} \end{aligned}$$

is a homomorphism. Moreover, for $x \in X(k)$

$$\phi_{t^*_x\mathcal{L}} = \phi_{\mathcal{L}}.$$

Now we take a look at abelian varieties over $\mathbb{C}$. From an analytical point of view, an abelian variety $X(\mathbb{C})$ is a compact complex Lie group of dimension g. If we take V to be the tangent space of $X(\mathbb{C})$ at the identity, then one knows from differential geometry that $\forall v \in V$, one has a unique diffeomorphism $\varphi_v : \mathbb{C} \longrightarrow X(\mathbb{C})$ with $d\varphi_v(0) = v$ and

$$\begin{aligned} \exp : V &\longrightarrow X(\mathbb{C}) \\ v &\longmapsto \varphi_v(1). \end{aligned}$$

Clearly, exp is a surjective group homomorphism since for all $z \in \mathbb{C}$ one has that $d\exp(zv)\exp(zw)_{|z=0} = v + w$. By uniqueness, one gets $\exp(zv)\exp(zw) = \exp z(v+w)$ and $\exp(V) \leq X(\mathbb{C})$ contains a neighbourhood of 1. If $\Gamma := \ker(\exp)$, then Γ is discrete (since exp is a local homéomorphism) and by compactness, it must have full rank. Thus, one obtains the analytic uniformisation of abelian varieties over $\mathbb{C}$:

$$X(\mathbb{C}) \cong {}^{\mathbb{C}^g}/_{\Gamma} \quad \text{where } \Gamma \text{ is a lattice in } \mathbb{C}^g. \tag{B.1}$$

The isomorphism is understood as an isomorphism of Lie groups. This makes it easier to understand the torsion points of a complex abelian variety $X(\mathbb{C})$.
As a group, ${}^{\mathbb{C}^g}/_{\Gamma} \cong \left({}^{\mathbb{R}}/_{\mathbb{Z}}\right)^{2g}$. Hence, for an integer n, if we denote the **n-torsion group** of $X(\mathbb{C})$ by

$$X[n] := \{x \in X(\mathbb{C}) \mid nx = 0\},$$

one clearly has

$$X[n] \cong \left({}^{\mathbb{R}}/_{\mathbb{Z}}\right)^{2g}[n] \cong \left({}^{\mathbb{R}}/_{\mathbb{Z}}[n]\right)^{2g} \cong \left(\mathbb{S}^1[n]\right)^{2g} \cong \left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)^{2g}. \tag{B.2}$$

Remark B.2.10. This result still holds for any algebraically closed field k of characteristic zero by the Lefschetz principle. Moreover, we will see later on that this also holds for algebraically closed fields of positive characteristic with $(n, \operatorname{char} k) = 1$ since for any divisor d of n, the isogeny $[d]$ is finite, étale and has degree d^{2g}. Thus $X[d]$ is an étale group scheme of rank d^{2g} for all divisor d. It must hence be $\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)^{2g}$.

Definition B.2.11 (k-isogenies)**.** Let $X(k)$ and $Y(k)$ be abelian varieties over k. An **isogeny** is a homomorphism of abelian varieties $f : X(k) \longrightarrow Y(k)$ such that $\ker(f)$ is finite. The rank of the finite group scheme $\ker(f)$ is called the **degree** of f.

We have a useful characterisation of isogenies as follow:

Proposition B.2.12. *Let $f : X(k) \longrightarrow Y(k)$ be a homomorphism of abelian varieties. Then the following are equivalent:*

- *f is an isogeny.*
- $\dim X(k) = \dim Y(k)$ *and f is surjective.*
- *f is finite, flat and surjective.*

Proof. See [Mil08a](Prop. 8.1). □

Example B.2.13. This makes it easier to exhibit some examples of isogenies.

(i) If $k = \mathbb{C}$, an isogeny is a covering $f : X(\mathbb{C}) \longrightarrow Y(\mathbb{C})$ with Galois group $\ker(f)$.

(ii) If k is any field of $\operatorname{char} k \geq 0$, then an isogeny is an étale map.

Let $X(k)$ be an abelian variety over k and let $\operatorname{End}_k(X) := \operatorname{Hom}_k(X, X)$ be its endomorphsim ring. Note that if $f \in \operatorname{End}_k(X)$ is not an isogeny, it cannot be surjective. Define for all $f, g \in \operatorname{End}_k(X)$

$$\begin{aligned} \deg_k \, : \operatorname{End}_k(X) &\longrightarrow \mathbb{N} \\ f &\longmapsto \deg_k(f) = \begin{cases} \operatorname{rank}\ker(f) & \text{if } f \text{ is an isogeny} \\ 0 & \text{else} \end{cases} \end{aligned}$$

with $\deg_k(fg) = \deg_k(f)\deg_k(g)$. A noticeable endomorphism of $X(k)$ is the map

$$\begin{aligned} [n] \, : X(k) &\longrightarrow X(k) \\ x &\longmapsto \underbrace{x + \cdots + x}_{n \text{ times}} \end{aligned}$$

Hence, we adopt the new notation $[-1] := inv$ for the inverse map and define $[-n] := [n] \circ [-1]$. This defines a canonical isomorphism $\mathbb{Z} \cong A \leq \operatorname{End}_k(X)$ where A is a subring of $\operatorname{End}_k(X)$. One then has that

$$X[n] = \ker\left([n] : X(k) \longrightarrow X(k)\right).$$

Suppose $\mathcal{M}$ is a very ample line bundle, then $\mathcal{L} := \mathcal{M} \times [-1]^*\mathcal{M}$ is ample and

$$[-1]^*\mathcal{L} \cong \mathcal{L} \qquad \text{(such a line bundle is called } \textit{symmetric}\text{)}$$

Now take $\mathcal{L}' := \mathcal{L}^{\otimes r}$ for some sufficiently large r so that $\mathcal{L}'$ is very ample. Then from (corollary B.2.8), one has

$$[n]^*\mathcal{L}' \cong \mathcal{L}'^{\otimes n^2}.$$

In particular, their restriction on $X[n]$ is trivial (since the map $X[n] \hookrightarrow X(k) \xrightarrow{[n]} X(k)$ is constant) which is only true if $X[n]$ has rank 0 and thus $[n]$ is an isogeny. Thus, for $\mathcal{L}' = \mathcal{O}_X(D)$ where D is an effective Cartier divisor, the intersection product is given by

$$\deg_k[n] \cdot D \ldots D = [n]^*D \ldots [n]^*D = n^2 D \ldots n^2 D = n^{2g} D \ldots D$$

As $D \ldots D \neq 0$ (since $\mathcal{L}'$ is very ample), one sees that

$$[n] \text{ is an isogeny of degree } \deg_k[n] = n^{2g}. \tag{B.3}$$

Remark B.2.14 (Isogeny as an equivalence relation). Finally, given an isogeny $f : X(k) \longrightarrow Y(k)$ of abelian varieties; of degree say n; one can construct an isogeny $g : Y(k) \longrightarrow X(k)$ such that $g \circ f = f \circ g = [n]$ as follow:

$$\begin{tikzcd}
X(k) \arrow[r, "\pi_1"] \arrow[rr, bend left, "f"] \arrow[dr, "\pi_2"'] \arrow[ddr, bend right, "{[n]}"'] & X(k)/_{\ker f} \arrow[r, "\psi", "\simeq"'] \arrow[d, "\pi"] & Y(k) \arrow[dl, dashed, "\varphi"] \arrow[ddl, dashed, bend left, "g"] \\
 & X(k)/_{X[n]} \arrow[d, "\rho", "\simeq"'] & \\
 & X(k) &
\end{tikzcd}$$

where

- ψ exists by universal property.
- $\varphi = \pi \circ \psi^{-1}$
- ρ exists by universal property.
- $g = \rho \circ \psi$

$g = \rho \circ \pi \circ \psi^{-1}$ is an isogeny since π is an isogeny and ψ and ρ are isomorphisms. This shows that isogeny induces an equivalence relation in the category of abelian varieties.

Let $\mathrm{Pic}^0(X) = \{$ line bundles $\mathcal{L}$ on $X(k) \mid t_x^*\mathcal{L} \cong \mathcal{L}\, \forall x \in X(k) \}$.

Definition B.2.15. Let $X(k)$ be an abelian variety over k. The **dual abelian variety** of $X(k)$ is defined to be the abelian variety $X^\vee(k)$ along with the line bundle $\mathcal{P}$ on $X(k) \times X^\vee(k)$ satisfying:

(i) $\mathcal{P}_{|X(k)\times\{y\}} \in \mathrm{Pic}^0(X_y)$.

(ii) $\mathcal{P}_{|\{0\}\times X^\vee(k)}$ is trivial.

(iii) The pair $(X^\vee(k), \mathcal{P})$ satisfies the following universal property: For all pair $(Y, \mathcal{M})$ where Y is an abelian variety and $\mathcal{M}$ a line bundle on $X \times Y$ satisfying (i) and (ii), there exists a unique morphism $f : Y \longrightarrow X^\vee(k)$ such that

$$\mathcal{M} \cong (id \times f)^*\mathcal{P}.$$

This is equivalent to

$$\mathrm{Hom}_k(Y, X^\vee) \leftrightarrow \left\{ \begin{array}{c} \text{line bundles } \mathcal{M} \text{ on } X \times Y \\ \text{satisfying (i) and (ii)} \end{array} \right\} \Big/_{\cong}.$$

In particular, if $Y = \mathrm{Spec}(k)$ then $X^\vee(k) = \mathrm{Pic}^0(X)$.

Mumford constructs the dual abelian variety scheme-theoretically in [DM70](III. §13) as follows: He shows first that for an abelian variety $X(k)$ and some arbitrary scheme S there exists a unique closed subscheme $S' \leq S$ such that for every scheme T and every morphism $f : Z \longrightarrow S$:

$$f \text{ factors through } S' \quad \Leftrightarrow \quad (\mathrm{id} \times f)^*\mathcal{M} \cong p_2^*\mathcal{L}' \text{ for some line bundle } \mathcal{L}' \text{ on } S,$$

where $\mathcal{M}$ is a line bundle on $X(k) \times S$. Applying this to the Mumford bundle on $X \times X$, one sees that

$$\mathcal{M}(\mathcal{L}) = m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} \quad \text{where } \mathcal{L} \text{ is a line bundle on } X(k).$$

This provides a closed subscheme X' with the same universal property. Since for each $x \in X(k)$

$$\mathcal{M}(\mathcal{L})_{|X(k)\times\{x\}} \cong t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

One can view $K(\mathcal{L})$ as a scheme, whose rational points are

$$K(\mathcal{L})(k) = \{x \in X(k) \,|\, \mathcal{M}(\mathcal{L})_{|X(k)\times\{x\}} \text{ is trivial } \} = X'(k)$$

Mumford then shows that it is a subgroup scheme of X, and that it is finite if and only if $\mathcal{L}$ is ample. (Recall that a line bundle $\mathcal{L}$ over a proper scheme S is *ample* if there exists a positive integer $n > 0$ such that $\mathcal{L}^{\otimes n}$ is basepoint-free and admits an embedding into projective space.) He thus defines the dual abelian variety $X^\vee(k)$ of $X(k)$ to be the quotient scheme

$$X^\vee(k) = {}^{X(k)}\big/_{K(\mathcal{L})(k)} \quad \text{for some ample line bundle } \mathcal{L} \text{ on } X(k).$$

One has a quotient map $\pi : X(k) \times X(k) \longrightarrow X(k) \times X^\vee(k)$ (induced by the action of $K(\mathcal{L})$) given by $\mathrm{id} \times \phi_{\mathcal{L}}$. It only suffices to show that the mumford bundle $\mathcal{M}(\mathcal{L})$ corresponds to a line bundle $\mathcal{P}$ on $X(k) \times X^\vee(k)$ such that $\pi^*\mathcal{P} = \mathcal{M}(\mathcal{L})$.

Moreover, Mumford shows that $X^\vee(k)$ is an abelian variety on k with the same dimension as $X(k)$ and that the duality is functorial in $X(k)$ with the property:

$$(X(k)^\vee)^\vee \cong X(k).$$

Definition B.2.16 (Polarisation)**.** Let $X(k)$ be an abelian variety. A **polarisation** of $X(k)$ is an isogeny $\varphi : X(k) \longrightarrow X^\vee(k)$ such that

$$\varphi_{|k} = \phi_{\mathcal{L}} \quad \text{for some very ample line bundle } \mathcal{L}.$$

where

$$\begin{aligned} \phi_{\mathcal{L}} \,:\, X(k) &\longrightarrow X^\vee(k) \\ x &\longmapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}. \end{aligned}$$

A polarization is said to be **principal** if it induces an isomorphism $X(k) \cong X^\vee(k)$.

Remark B.2.17. (i) Note that for $\dim X > 1$, $X(k)$ and $X^\vee(k)$ are isogenous but not necessarily isomorphic. Indeed, suppose $\operatorname{char} k = 0$ and let $X(k)$ be an abelian variety over k with $\operatorname{End}_k(X) \cong \mathbb{Z}$ and $\dim X = 2$. Let P be a point of order 2, and let $Y = {}^{X}\big/_{\langle P \rangle}$.

Suppose $f : Y \xrightarrow{\sim} Y^\vee$ was an isomorphism and consider the endomorphism

$$g : X \xrightarrow{\pi} Y \xrightarrow{f} Y^\vee \xrightarrow{\pi^\vee} X^\vee$$

Note that

$$\deg_k(g) = \deg_k(\pi)\deg_k(f)\deg_k(\pi^\vee) = \deg_k(q)^2 = 4.$$

But since $\mathrm{End}_k(X) \cong \mathbb{Z}$, there is an integer n such that $g = [n]$ (by remark B.2.14). Thus, $4 = \deg(g) = \deg([n]) = n^4$ (by B.3) which is impossible. Thus for $\dim X > 1$, a principal polarization does not need to exist.

More generally, if $X(k)$ is a g-dimensional p.p.a.v. over $\mathbb{C}$ with $\mathrm{End}_k(X) \cong \mathbb{Z}$, and G is a finite subgroup such that $|G| \neq n^g$ for some integer n then $Y(k) := {}^{X(k)}/_{G}$ is an abelian variety that admits no* principal polarization.

(ii) The morphism $\pi^\vee$ in (i) is called the *dual morphism* of π. More generally, for any morphism $f : X(k) \longrightarrow Y(k)$ of abelian varieties, there exists a *dual morphism* $f^\vee : Y^\vee(k) \longrightarrow X^\vee(k)$ by applying the universal property (iii) in definition B.2.15 with $\mathcal{M} = (f \times id)^*\mathcal{P}$, $\mathcal{P}$ the poincaré bundle on $Y \times Y^\vee$. Moreover, if f is an isogeny, then so is $f^\vee$ and one has $\deg_k(f) = \deg_k(f^\vee)$ (see [DM70] II. §15 Theorem 1)

B.3. Elliptic curves over local fields

In this section, we let E be an elliptic curve over some non-archimedean local field K. Let $\mathcal{O}_K$ its ring of integers, $\mathfrak{m}$ its maximal ideal and $k := {}^{\mathcal{O}_K}/_{\mathfrak{m}}$ its residue field (which is perfect). (Note that the results proven in this section are also valid for $K = \mathbb{C}$)

B.3.1. The Good, the Bad, and the Multiplicative

An elliptic curve $E(K)$ is given by a Weierstrass equation

$$E : y^2 + (a_1x + a_3)y = x^3 + a_2x^2 + a_4x + a_6 \quad \text{where } a_i \in \overline{K}.$$

We define the following useful quantities:

$$c_4 := (a_1^2 + 4a_2)^2 - 24(a_1a_3 + 2a_4) \quad \text{and} \quad c_6 := \begin{array}{l} -(a_1^2+4a_2)^3 + 36(a_1^2+4a_2)(a_1a_3+2a_4)) \\ \qquad -216(a_3^2+4a_6) \end{array}$$

And thus

$$\Delta(E) = \frac{c_4^3 - c_6^2}{1728} \quad \text{and} \quad j(E) = \frac{c_4^3}{\Delta(E)}.$$

*Although:

Theorem B.2.18 ([DM70] Corollary 1 p.216). *Every abelian variety $X(k)$ is isogenous to a p.p.a.v.*

One notices that any change of variable in (x,y) preserving this form of equation (i.e. changes of the form $(x' = u^{-2}x, y' = u^{-3}y)$ for some $u \in \overline{K}^{\times}$) produces coefficients $a_i u^i$. Hence, one can find a suitable u so that all coefficients of the given Weierstrass equation are in $\mathcal{O}_K$ (by choosing $\pi^n \mid u$ for some large n) and thus, $\Delta(E) \in \mathcal{O}_K$ and $v(\Delta) \geq 0$. (In fact there always exists a change of coordinates giving $c_4' = u^{-4}c_4, c_6' = u^{-6}c_6$ and $\Delta' = u^{-12}\Delta$)

Definition B.3.2. (Weierstrass model) The $\mathcal{O}_K$-scheme $X \hookrightarrow \mathbb{P}^2_{\mathcal{O}_K} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ defined by a Weierstrass equation with coefficients in $\mathcal{O}_K$ is said to be a **Weierstrass model** for $E(K)$. Moreover, if $v(\Delta)$ is minimal, then the model is called **minimal**. The fibre

$$\widetilde{E} : X \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} k$$

is called the **reduction** of E at v.

In other words, the curve $\widetilde{E}(K)$ defined by the minimal Weierstrass equation

$$\widetilde{E} : y^2 + (\widetilde{a}_1 x + \widetilde{a}_3)y = x^3 + \widetilde{a}_2 x^2 + \widetilde{a}_4 x + \widetilde{a}_6 \qquad \text{with } \widetilde{a}_i \in \mathcal{O}_K$$

is the *reduction* of $E(k)$. It is a genus 1 projective curve that always exists and is unique up to a change of coordinates. Moreover, it can only have at most one singularity: (In order to avoid computational complications -this is a fancy way to just say I am lazy- we may assume that $\operatorname{char} k \geq 5$)
Any projective variety X given by a Weierstrass equation has the form

$$X : y^2 = x^3 + Ax + B.$$

At any singular point (x_0, y_0) one has that

$$2y_0 = 3x_0^2 + A = 0 \quad \Rightarrow \quad y_0 = 0$$

and x_0 is a double root of $x^3 + Ax + B$, hence there can only be one.

Definition B.3.3. (Type of reduction of an elliptic curve) Let E be an elliptic curve as above, $\widetilde{E}(k)$ its reduction at some v.

(i) $E(k)$ is said to have **good** reduction if $\widetilde{E}(k)$ is non singular (i.e. an elliptic curve over k.)

(ii) If not, then $E(k)$ is said to have **bad** reduction. We distinguish two cases:
   (a) if $\widetilde{E}(k)$ has a node, we say that $E(k)$ has **multiplicative** reduction.
   (b) if $\widetilde{E}(k)$ has a cusp, we say that $E(k)$ has **additive** reduction.

In practice, the elliptic curve $E(K)$ has good reduction if and only if the discriminant Δ is a unit in $\mathcal{O}_K$. When $v(\Delta) > 0$, the reduction is multiplicative if $\widetilde{c}_4 \in \mathcal{O}_K^{\times}$ and is additive if $v(\widetilde{c}_4) = 0$.

Example B.3.4. For $K = \mathbb{Q}$, no elliptic curve can have everywhere good reduction. Indeed, if it did, that would mean that the equation

$$4A^3 + 27B^2 = \pm 1$$

would have solutions in $\mathbb{Q}$ for some integers A, B, which is just... not true.

However, an elliptic curve that has bad reduction over K can have good reduction over some of its finite extensions. When this is the case, $E(K)$ is said to have **potential good** reduction over K. We have the following useful criteria:

Proposition B.3.5. *An elliptic curve $E(K)$ has potential good reduction if and only if $j(E) \in \mathcal{O}_K$.*

Proof. For convenience (here is another one), suppose $\operatorname{char} K \geq 5$ and consider an elliptic $E'(K')$, with K' a finite extension of K.

⇒) Suppose E' has good reduction over K'. Then its discriminant $\Delta' \in \mathcal{O}_{K'}^\times$ and thus

$$j(E) = \frac{(\tilde{c}_4')^3}{\Delta'} \in \mathcal{O}_{K'}^\times$$

where $\tilde{c}_4'$ is as seen above. Since $E(K)$ is defined over K, $j(E) \in K$ and hence $j(E) \in \mathcal{O}_K$.

⇐) Let K' be some finite extension of K such that E admits a Weierstrass equation of the form†:

$$E : y^2 = x(x-1)(x-\lambda) \quad \text{with } \lambda \in K', \lambda \neq 0, 1.$$

Observe that, since $j(E) \in \mathcal{O}_K$

$$(1 - \lambda + \lambda^2)^3 - j(E)\lambda^2(1-\lambda)^2 = 0 \quad \Rightarrow \quad \lambda \in \mathcal{O}_K \quad \text{and} \quad \lambda \not\equiv 0, 1 \mod \mathfrak{m}$$

Hence E has integral coefficients and potential good reduction.

□

Example B.3.6.
- An elliptic curve $E(\mathbb{C})$ with CM has everywhere potential good reduction. More generally, over a number field $\mathbb{F}$ with CM, $E(\mathbb{F})$ has potential good reduction at every prime of $\mathbb{F}$.
- We illustrate an explicit example: Let $\mathbb{F} := \mathbb{Q}_5$ and consider the elliptic curve given by

$$E : y^3 = x^3 + 5 \tag{B.4}$$

Then clearly one has

$$c_4 = 0 \quad \text{and} \quad \Delta(E) = -10800 = -2^4 \cdot 3^3 \cdot 5^2 \quad \Rightarrow \quad \begin{cases} v_5(c_4) = \infty & > 0, \\ v_5(\Delta) = 2 & > 0. \end{cases}$$

†This is called the *Legendre form* and always exists, provided $\operatorname{char} K \neq 2$. See ([Sil86] III, 1.7).

Hence, $E(\mathbb{Q}_5)$ has (additive) bad reduction on $\mathbb{Q}_5$.
Now, consider the the finite extension $\mathbb{Q}_5(\sqrt[6]{5})/\mathbb{Q}_5$, then the equation (B.4) is no longer minimal. We consider the change of variable $\left(x' = \left(5^{-\frac{1}{3}}\right)x, y' = \left(5^{-\frac{1}{2}}\right)y\right)$, we then obtain the equation

$$E' : (y')^3 = (x')^3 + 1$$

We clearly see that

$$\Delta' := \Delta(E') = -432 = -2^4 \cdot 3^3 \quad \Rightarrow \quad v_5(\Delta') = 0$$

and thus, $E(\mathbb{Q}_5(\sqrt[6]{5}))$ has good reduction.

B.3.7. L-functions associated to elliptic curves

Let E be an elliptic curve over a number field $\mathbb{F}$. Let $\mathfrak{p}$ be a prime of $\mathbb{F}$ and $\mathbb{F}_\mathfrak{p}$ be the residue field at $\mathfrak{p}$, with $q_{v_\mathfrak{p}}$ its number of elements. We define the polynomial

$$f_v(X) = \begin{cases} q_v X^2 - a_v X + 1 & \text{if} \quad E \text{ has good reduction at } v \\ \pm X + 1 & \text{if} \quad E \text{ has (multiplicative) bad reduction at } v \\ 1 & \text{else} \end{cases}$$

where $a_v := q_v + 1 - \#\widetilde{E}(\mathbb{F}_v)$.

Definition B.3.8 (Weil-Hesse L-function)**.** For an elliptic curve E over a number field $\mathbb{F}$, The **Hasse-Weil** L-function of E is defined to be the Euler product

$$L(E(\mathbb{F}), s) = \prod_{v \nmid \infty} \frac{1}{L_v(E(\mathbb{F}), s)}$$

where, the factors are called the **local L-functions** at v and are defined by

$$L_v(E(\mathbb{F}), s) := f_v\left(\frac{1}{q_v^s}\right).$$

This product converges for values $\Re(s) > \frac{3}{2}$.

Remark B.3.9. (i) For the places at infinity, J.-P. Serre gave (relatively out of the scope of this thesis) definition in the *Séminaire Delange-Pisot-Poitou* in 1970, this can be found in [Ser70].

(ii) A CM elliptic curve over a number field $\mathbb{F}$ cannot have multiplicative reduction: Given an elliptic curve $E(\mathbb{F})$ and a (finite) place v. If $E(\mathbb{F})$ has CM by $\mathcal{O}_K$ with K an imaginary quadratic field, then as seen before, it has potentially good reduction everywhere. Let $\mathbb{F}'$ be a field extension where it has good reduction, v' a valuation extending v. Consider the change of variable $(x' = f(u,x), y' = g(u,y))$ such that E is minimal on $\mathbb{F}'$ with $c_4' = u^{-4}c_4$, $\Delta' = u^{-12}\Delta$.
As $u \in \mathcal{O}_{\mathbb{F}'}$, one has that

$$v'(c_4') + 4v'(u) = v'(c_4) \geq 0 \quad \text{and} \quad 0 \leq v'(\Delta') + 4v'(u) = v'(\Delta') \geq 0$$

Hence,

$$0 \leq v'(u) \leq \min\left(\frac{1}{12}v'(\Delta), \frac{1}{4}v'(c_4)\right).$$

But E has good reduction on $\mathbb{F}'$, so $v(\Delta) = 0$ and thus, $v'(u) = 0$. One finally gets that $v'(\Delta') = v'(\Delta) = 0$ and this implies good reduction at $\mathbb{F}'$.
This shows that the CM elliptic curve $E(\mathbb{F})$ cannot have multiplicative reduction, so either good or (additive) bad. Hence, its local L-function is given by the quadratic polynomial

$$f_v(X) = \begin{cases} q_v X^2 - a_v X + 1 = (1 - \alpha_v X)(1 - \overline{\alpha_v} X) & \text{if} \quad E \text{ has good reduction at } v \\ 1 & \text{if} \quad E \text{ has bad reduction at } v \end{cases}$$

We end this section by the following: The key in relating the local L-functions to the Hecke ones is the observation: one might think of the function $v \longmapsto \alpha_v$ as an algebraic Hecke character on K, with its values in $\mathbb{F}$. This would allow to write the L-function of a CM Elliptic curve as the product of two Hecke L-functions of its Hecke character:

Theorem B.3.10 (Deuring)**.** *Let E be an elliptic curve over $\mathbb{F}$ with CM by $\mathcal{O}_K$ and suppose that $K \subset \mathbb{F}$. Then the global L-series of $E(\mathbb{F})$ is given by*

$$L(E(\mathbb{F}), s) = L(s, \chi_E)L(s, \overline{\chi}_E))$$

Proof. For a proof, see [Sil94]. □

List of notations/Index

z, z'	complex numbers
$\mathfrak{I}(z)$	imaginary part of z
$\mathfrak{R}(z)$	real part of z
$\varepsilon_n(z)$	Eisenstein's trigonometric function $\sum_{k=-\infty}^{+\infty} \frac{1}{(z+k)^n}$
ω_1, ω_2	a pair of fundamental periods
Γ	a lattice $\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ in $\mathbb{C}$
u, v	some generators of Γ
$A(\Gamma)$	area of Γ divided by π
ξ	change of variable $\xi = \frac{z}{u}$
τ	modular period
B_{2n}	Bernoulli numbers
$E_n(z;\Gamma)$	Eisenstein series $\sum_{\gamma\in\Gamma} \frac{1}{(z+\gamma)^k}$
$\sum_e$	Eisenstein's summation
e_{2k}	the value of $E_{2k}(z;\Gamma) - \frac{1}{z^{2k}}$ near 0.
$\zeta(z;\Gamma)$	Weierstrass's zeta function
$\wp(z;\Gamma)$	Weierstrass's p function
$\sigma(z;\Gamma)$	Weierstrass's sigma function
$E_n^*(z;\Gamma)$	modified Eisenstein series
$E_{a,b}(z;\Gamma)$	Eisenstein-Kronecker series $\sum_{\gamma\in\Gamma} \frac{(\overline{z}+\overline{\gamma})^a}{(z+\gamma)^b}$
$E_{a,b}^*(z;\Gamma)$	modified Eisenstein-Kronecker series
$e_{a,b}$	the values of $E_{a,b}(z) - \frac{\overline{z}^a}{z^b}$ near $z = 0$
$e_{a,b}^*$	the values of $E_{a,b}^*(z) - \frac{\overline{z}^a}{z^b}$ near $z = 0$
$\langle\,.\,,.\,\rangle_\Gamma$	a complex pairing defined by $\langle\, z, z'\,\rangle_\Gamma = \exp\left(\frac{z\overline{z'} - z'\overline{z}}{A(\Gamma)}\right)$
$\sum_{\gamma\in\Gamma}^{*}$	sum over $\gamma \in \Gamma$ with $\gamma \neq -z$ when $z \in \Gamma$
$K_a^*(z, z';\Gamma)$	Eisenstein-Kronecker-Lerch series $\sum_{\gamma\in\Gamma}^{*} \frac{(\overline{z}+\overline{\gamma})^a}{\lvert z+\gamma\rvert^{2s}} \langle\,\gamma, z'\,\rangle_\Gamma$
$\theta(z;\Gamma)$	Jacobi theta function $\sum_{\gamma\in\Gamma} e^{-\pi z(\gamma\cdot\gamma)}$
$\theta_{a,t}(z, z')$	modified Jacobi theta function $\sum_{\gamma\in\Gamma} e^{-t\lvert z+\gamma\rvert^2} (\overline{z} + \overline{\gamma})^a \langle\,\gamma, z'\,\rangle$
$e_{a,b}^*(\Gamma)$	Eisenstein-Kronecker numbers $e_{a,b}^*(0,0;\Gamma) = K_{a+b}^*(0,0,b)$.
K, $\mathcal{O}_K$	a number field K and its ring of integers $\mathcal{O}_K$
K_v, $\mathcal{O}_v$	completion of K with respect to the valuation v, its ring of integers $\mathcal{O}_v$;
π_v	uniformizer of $\mathcal{O}_v$
χ	Hecke (resp. Dirichlet) character
χ_∞	infinity type of a Hecke character χ
$\mathfrak{f}$	conductor of a Hecke character

$I(\mathfrak{f})$	set of fractional ideals of K prime to $\mathfrak{f}$
$P(\mathfrak{f})$	set of principal fractional ideals of K prime to $\mathfrak{f}$
$P_{\mathfrak{f}}$	set of principal fractional ideals of $\mathfrak{a} = (\alpha)$ such that $\alpha \equiv 1 \bmod^* \mathfrak{f}$
$Cl(K)$	ideal class group of K
$\mathbb{A}_K$	ring of adeles of K
$\mathbb{J}_K$	group of ideles of K
$\mathfrak{Cl}(K)$	the idele class group $\left(\mathbb{J}_K / K^\times\right)$ of K
$\mathscr{O}_X$	sheaf on X
$\mathcal{L}$	line bundle
t_z	translation operator by z
e_γ	multiplier i.e. holomorphic invertible function satisfying (4.1)
ϑ_s	reduced theta function associated to a section s of a line bundle $\mathcal{L}$
ϑ_D	reduced theta function associated to a divisor with associated line bundle $\mathcal{L} \cong \mathscr{O}_X(D)$
ϑ_D^{+w}	translated theta function (4.6)
α_z	character defined by $\alpha_z(\gamma) = \exp\left(2\pi i E(v,\gamma)\right)$
$\mathrm{Pic}(X)$	Picard group of isomoprhism classes of line bundles on X
$\mathrm{Pic}^0(X)$	connected component of $\mathrm{Pic}(X)$
$\mathcal{P}$	the Poincare bundle
θ	theta function associated to the divisor $D = [0]$ given by $\theta(z) = \exp\left(-\frac{e_2^*}{2} z^2\right)\sigma(z)$
Θ	the Kronecker theta function given by $\Theta(z,z') = \frac{\theta(z+z')}{\theta(z)\theta(z')}$
$\Theta_{(w,w')}$	translated Kronecker theta function (4.13)
$A(\mathbb{F})$	(analytic) abelian variety over $\mathbb{F}$
g	dimension of A
$X(\mathbb{F})$	(algebraic) abelian variety over $\mathbb{F}$
$A[N]$	N-torsion group of A
$T_l(A)$	l-adic Tate module of A
$\widehat{T}(A)$	Adelic Tate module of A
$\overset{\mathcal{M}}{\Theta}_{(w,w')}$	algebraic Kronecker Theta function (5.8)

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